THE HODGE–TATE PERIOD MAP

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Contents

1. Introduction

1.1. **Overview.** The main result is the following.

Theorem 1.1.1. For any tame level $K^p \subset \mathrm{GSp}_{2g}(\mathbb{A}_f^p)$ contained in $\{\gamma \in \mathrm{GSp}_{2g}(\widehat{\mathbb{Z}}^p) \mid \gamma \equiv 1 \bmod N\}$ for some $N \geq 3$ prime to p, there exists a perfectoid space $\mathcal{X}^*_{\Gamma(p^\infty),K^p}$ over $\mathbb{Q}_p^{\mathrm{cycl}}$ such that

$$\mathcal{X}^*_{\Gamma(p^\infty),K^p} \sim \lim_m \mathcal{X}^*_{\Gamma(p^m),K^p}.$$

Moreover, there is a $\mathrm{GSp}_{2g}(\mathbb{Q}_p)$-equivariant Hodge–Tate period map

$$\pi_{\mathrm{HT}} : \mathcal{X}^*_{\Gamma(p^\infty),K^p} \to \mathscr{F}\ell.$$

The strategy is to construct the map π_{HT} in steps.

First we construct a map of the underlying topological spaces

$$|\pi_{\mathrm{HT}}| : |\mathcal{X}^*_{\Gamma(p^\infty)}| \backslash |\mathcal{Z}_{\Gamma(p^\infty)}| \to |\mathscr{F}\ell|,$$

which is constructed using the moduli interpretation of Shimura varieties and the Hodge–Tate filtration.

1.2. **Notations.** Throughout this paper, $0 \leq \epsilon < 1/2$ is a number such there exists an element in $\mathbb{Z}_p^{\mathrm{cycl}}$ of valuation ϵ, and any such element will be denoted by $p^\epsilon \in \mathbb{Z}_p^{\mathrm{cycl}}$. We also assume that $g \geq 2$.

Definition 1.2.1. Fix an element $t \in (\mathbb{Z}_p^{\mathrm{cycl}})^\flat$ such that $|t| = |t^\sharp| = |p|$, such that t admits a $(p-1)$-th root. Then we get an identification $(\mathbb{Z}_p^{\mathrm{cycl}})^\flat = \mathbb{F}_p[[t^{1/(p-1)p^\infty}]]$.

2. Technical tools

2.1. Canonical subgroups.

Definition 2.1.1. Let $A \to S$ be an Abelian scheme with S of characteristic $p > 0$. Let $e : S \to A$ be the unit section. Let $\omega_{A/S}$ be the line bundle on S defined as $\wedge^g e^* \Omega^1_{A/S}$. The Verschiebung map $V : A^{(p)} \to A$ induces a map $\omega_{A/S} \to \omega_{A^{(p)}/S} \simeq \omega_{A/S}^{\otimes p}$, which in turn induces a canonical section $\mathrm{Ha}(A/S) \in H^0(S, \omega_{A/S}^{\otimes(p-1)})$, called the Hasse invariant of A/S.

Definition 2.1.2. Let R be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let $A \to \mathrm{Spec}(R)$ be an Abelian scheme. Let $A_1 \to \mathrm{Spec}(R_1)$ be its reduction modulo p, where $R_1 = R/p$. For an integer $m \geq 1$, the Abelian scheme $A \to \mathrm{Spec}(R)$ is said to satisfy the weak $O(m, \epsilon)$ condition if

$$\mathrm{Ha}(A_1/\mathrm{Spec}(R_1))^{(p^m-1)/(p-1)} \in H^0(R_1, \omega^{\otimes(p^m-1)})$$

divides p^ϵ, in the sense that there exists $u \in H^0(R_1, \omega^{\otimes(1-p^m)})$ such that $u \cdot \mathrm{Ha}(A_1/\mathrm{Spec}(R_1))^{(p^m-1)/(p-1)} = p^\epsilon$ as elements in $R_1 = R/p$.

The Abelian scheme $A \to \mathrm{Spec}(R)$ is said to satisfy the strong $O(m, \epsilon)$ condition if $\mathrm{Ha}(A_1/R_1)^{p^m}$ divides p^ϵ.

Lemma 2.1.3. Let S be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let G be a finite locally free commutative group scheme over S. Let $C_1 \subset G \otimes_S S/p$ be a finite locally free subgroup. Assume that for $H = (G \otimes_S S/p)/C_1$, multiplication by p^ϵ on the Lie complex $\ell_H^\vee$ is homotopical to zero. Then there exists a finite locally free subgroup $C \subset G$ over S such that $C \otimes_S S/p^{1-\epsilon} = C_1 \otimes_{S/p} S/p^{1-\epsilon}$.

Proof. We will apply Lemma A.0.9. Take $A = S/p$, $B = S/p^{2-\epsilon}$, and

$$B' = \{(x, y) \in S/p^{2-2\epsilon} \times S/p \mid x = y \in R/p^{1-\epsilon}\}.$$

The map $B \to B'$ is given by $x \mapsto (x, x)$. Let J (resp. J') be the kernel of $B \to A$ (resp. $B' \to A$). Then both J and J' are isomorphic to $S/p^{1-\epsilon}$ as Abelian groups. The transition map $S/p^{1-\epsilon} \simeq J \to J' \simeq S/p^{1-\epsilon}$ is given by multiplication by p^ϵ. Let K be the cone of the map $\ell_{C_1}^\vee \to \ell_{G\otimes_S S/p}^\vee$ of Lie complexes. Then $K \simeq \ell_H^\vee$ by Remark A.0.6. In particular, multiplication by p^ϵ is homotopic to zero on K. Then the image of the obstruction $o \in \mathrm{Ext}^1(H, K \otimes^L J)$ in $\mathrm{Ext}^1(H, K \otimes^L J')$ is zero. The vanishing of the obstruction immediately shows the existence of a lift $C \subset G$ such that $C \otimes_S S/p^{1-\epsilon} = C_1 \otimes_{S/p} S/p^{1-\epsilon}$. □

Lemma 2.1.4. Let R be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let X be a scheme over R such that $\Omega^1_{X/R}$ is killed by p^ϵ for some $\epsilon > 0$. Then the map $X(R) \to X(R/p^\delta)$ is injective for all $\delta > \epsilon$.

Proof. Omitted. □

Lemma 2.1.5. Let R be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let $A \to \mathrm{Spec}(R)$ be an Abelian scheme satisfying weak $O(m, \epsilon)$. Then there is a unique closed subgroup $C_m \subset A[p^m]$ such that $C_m = \ker(F^m)$ mod $p^{1-\epsilon}$.

Proof. Let $H_1 = \ker(V^m : A_1^{(p^m)} \to A_1)$ be the kernel of the m-th composition of the Verschiebung map. We have a short exact sequence $0 \to H_1 \to A_1^{(p^m)} \to A_1 \to 0$. Taking the Lie complex of each term, we see that $\ell_{H_1}^\vee$ is represented by the complex $\mathrm{Lie}(A_1^{(p^m)}) \to \mathrm{Lie}(A_1)$. Note that the determinant of $\mathrm{Lie}(A_1^{(p^m)}) \to \mathrm{Lie}(A_1)$ is simply

$$\mathrm{Ha}(A_1/R_1)^{(p^m-1)/(p-1)} \in H^0(R_1, \omega^{\otimes(p^m-1)}),$$

which is a direct corollary of the definition of the Hasse variant. It follows that multiplication by $\mathrm{Ha}(A_1/R_1)^{(p^m-1)/(p-1)}$ on $\ell_{H_1}^\vee$ is null-homotopic. As the Abelian scheme A satisfies the weak $O(m, \epsilon)$ condition, we conclude that multiplication by p^ϵ on $\ell_{H_1}^\epsilon$ is null-homotopic. Thus Lemma 2.1.3 shows the exsitence of $C_m \subset A[p^m]$ such that

$$C_m \otimes_R R/p^{1-\epsilon} = \ker(F^m) \otimes_{R_1} R/p^{1-\epsilon}.$$

To show that the subgroup C_m is unique, we will directly describe the points of C_m: for every p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra R' with $R \to R'$, we have

$$C_m(R') = \{s \in A[p^m](R') \mid s \equiv 0 \bmod p^{(1-\epsilon)/p^m}\}.$$

It sufficies to prove the equality for $R' = R$.

Let $s \in C_m(R)$. Since $C_m \otimes_R R/p^{1-\epsilon} = \ker(F^m) \otimes_{R_1} R/p^{1-\epsilon}$, the image of s in $A(R/p^{1-\epsilon}) = A_1(R/p^{1-\epsilon})$, denoted by $s_{1-\epsilon}$, lies in the kernel of F^m. Thus $s_{1-\epsilon}$ lies in the kernel of $A_1(R_{1-\epsilon}) \to A_1(\mathrm{Fr}_*^m R_{1-\epsilon})$, where Fr is the absolute Frobenius. Note that $s_{1-\epsilon}$ also lies in $A_1[p^m](R_{1-\epsilon})$. Hence $s \equiv 0 \bmod p^{(1-\epsilon)/p^m}$.

Before we prove the converse, we need the following result. Since multiplication by p^ϵ is null-homotopic on $\ell_{H_1}^\vee$, we see that p^ϵ kills $\mathrm{Lie}(H_1)^\vee = e^*\Omega_{H_1/R_1}$. Thus p^ϵ kills $\Omega^1_{H_1/R_1}$. Let $H = A[p^m]/C_m$. Note that $H \otimes_R R_{1-\epsilon} = H_1 \otimes_{R_1} R_{1-\epsilon}$. Hence $\Omega^1_{H \otimes_R R_{1-\epsilon}/R_{1-\epsilon}} \simeq \Omega^1_{H/R}/p^{1-\epsilon}$ is killed by p^ϵ. Since $\Omega^1_{H/R}$ is p-adically complete, it follows that $\Omega^1_{H/R}$ is killed by the multiplication by p^ϵ map.

Now let $s \in A[p^m](R)$ be an element such that $s \equiv 0 \bmod p^{(1-\epsilon)/p^m}$. By a similar argument as above, we conclude that $s_{1-\epsilon} \in C_m(R/p^{1-\epsilon}) \subset A[p^m](R/p^{1-\epsilon})$. Then the image $t \in H(R)$ of s is 0 modulo $p^{1-\epsilon}$. Finally, apple Lemma 2.1.4 with $\delta = 1 - \epsilon$, we conclude that $t = 0 \in H(R)$, showing that $s \in C_m(R)$ as desired. □

Definition 2.1.6. Let R be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. We say that an Abelian scheme $A \to \mathrm{Spec}(R)$ has a weak canonical subgroup of level m if $A \to \mathrm{Spec}(R)$ satisfies weak $O(m, \epsilon)$ for some $\epsilon < 1/2$. In that case, we call $C_m \subset A[p^m]$ in Lemma 2.1.5 the weak canonical subgroup of level m.

If moreover A satisfies the strong $O(m, \epsilon)$ condition, then we say that C_m is a strong canonical subgroup.

Lemma 2.1.7. Let R be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let A and B be Abelian schemes over R.

(1) If A has a canonical subgroup $C_m \subset A[p^m]$ of level m, then it has a canonical subgroup $C_{m'} \subset A[p^{m'}]$ of every level $m' \leq m$, and $C_{m'} \subset C_m$.
(2) Let $f : A \to B$ be a map of Abelian schemes. Assume that both A and B have canonical subgroups $C_m \subset A[p^m]$ and $D_m \subset B[p^m]$ of level m. Then C_m maps into D_m under f.
(3) Assume that A has a canonical subgroup $C_m \subset A[p^m]$ of level m, and let $\overline{x}$ be a geometric point of $\mathrm{Spec}(R[p^{-1}])$. Then $C_m(\overline{x}) \simeq (\mathbb{Z}/p^m\mathbb{Z})^g$, where g is dimension of the Abelian variety over $\overline{x}$.

Proof. Omitted. □

2.2. **Hartog's extension principle.** Let's recall Hartog's theorem of analytic functions.

Theorem 2.2.1 (Hartog's Theorem)**.** Let $G \subset \mathbb{C}^n$ be an open subset with $n \geq 2$, and let K be a compact subset of G. If $G \backslash K$ is connected, then any holomorphic function on $G \backslash K$ can be extended to a holomorphic function on G in a unique way.

We shall establish several analogies of Hartogs' theorem.

Lemma 2.2.2 ([GR68, Lemma III.3.1, Proposition III.3.3])**.** Let X be a locally Noetherian scheme. Let $Z \subset X$ be a closed subscheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $n \geq 1$ be an integer. Then the following are equivalent:

(1) For any open subscheme V of X, the map
$$H^i(V, \mathcal{F}) \to H^i(V \backslash Z, \mathcal{F})$$
is bijective for $i \leq n-2$ and injective for $i = n-1$.
(2) For any open subscheme V of X, the local cohomology
$$H^i_{V \cap Z}(V, \mathcal{F}) = 0$$
for all $i \leq n-1$.
(3) For any $x \in Z$ the depth of $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module is at least n.

Lemma 2.2.3 (Serre's criterion)**.** A Noetherian ring R is normal if and only if $R_\mathfrak{p}$ is regular for every $\mathfrak{p}$ of height ≤ 1 and $R_\mathfrak{p}$ has depth ≥ 2 for every $\mathfrak{p}$ of height ≥ 2.

Lemma 2.2.4. Let R be a normal ring, i.e. the localization $R_\mathfrak{p}$ is an integrally closed domain for every prime ideal $\mathfrak{p}$ of R. Assume R is Noetherian. Let $Z \subset \mathrm{Spec}(R)$ be a closed subscheme of codimension at least 2, i.e. every $\mathfrak{p} \in Z$ has height at least 2. Then for $U = \mathrm{Spec}(R) \backslash Z$,
$$H^0(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)}) \simeq H^0(U, \mathcal{O}_{\mathrm{Spec}(R)}).$$

Proof. Consider $n = 2$ and $\mathcal{F} = \mathcal{O}_X$ in Lemma 2.2.2. Serre's criterion, cf. Lemma 2.2.3, guarantees the third condition in Lemma 2.2.2. The first assertion gives the desired result. □

It can also be proved directly as follows.

Lemma 2.2.5. Let X be a locally Noetherian normal scheme. Let U be an open subscheme of X with codimension ≥ 2. Then the map $H^0(X, \mathcal{O}_X) \to H^0(U, \mathcal{O}_X)$ is an isomorphism.

Proof. We may assume that $X = \mathrm{Spec}(A)$ where A is normal integral domain. For every non-empty open V of X, the ring $\Gamma(V, \mathcal{O}_X)$ may be considered as a subring of the function field $K(X) = \mathrm{Frac}(A)$ such that the restriction maps are given by inclusions of rings. Let Z be an irreducible closed subset of X of codimension 1. Then U intersects Z non-trivially, so it contains the generic point η of Z. In other words, the subring $\Gamma(U, \mathcal{O}_X)$ of the function field $K(X)$ is contained in the stalk $\mathcal{O}_{X,\eta}$. But $A = \Gamma(X, \mathcal{O}_X)$ is the intersection of all the stalks $\mathcal{O}_{X,\eta}$, where η is a prime ideal of height 1; in other words, where η is the generic point of an irreducible closed subset of codimension 1. ☐

Lemma 2.2.6. Let R be a topologically finitely generated, flat, and p-adically complete $\mathbb{Z}_p$-algebra, such that $\overline{R} = R/p$ is normal. Fix $f \in R$ such that its reduction $\overline{f} \in \overline{R}$ is not a zero-divisor. Let $0 < \epsilon \leq 1$. Set $S = (R \,\widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}})\langle u \rangle/(u \cdot f - p^\epsilon)$. Then S is p-adically complete and flat over $\mathbb{Z}_p^{\mathrm{cycl}}$. Fix a closed subscheme $Y \subset \mathrm{Spec}(\overline{R})$ of codimension ≥ 2. Let Z be the inverse image of Y in $\mathrm{Spf}(S)$. Then for $U = |\mathrm{Spf}(S)| \backslash Z$,

$$S = H^0(\mathrm{Spf}(S), \mathcal{O}_{\mathrm{Spf}(S)}) \simeq H^0(U, \mathcal{O}_{\mathrm{Spf}(S)}).$$

Proof. We first show that the map

$$S \simeq H^0(\mathrm{Spf}(S), \mathcal{O}_{\mathrm{Spf}(S)}) \to H^0(U, \mathcal{O}_{\mathrm{Spf}(S)})$$

is injective. Since S is p-adically separated and $H^0(U, \mathcal{O}_{\mathrm{Spf}(S)})$ is flat over $\mathbb{Z}_p^{\mathrm{cycl}}$, it suffices to show that

$$S_\epsilon \simeq H^0(\mathrm{Spec}(S_\epsilon), \mathcal{O}_{\mathrm{Spec}(S_\epsilon)}) \to H^0(U_\epsilon, \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$$

is injective, where $S_\epsilon = S/p^\epsilon$, Z_ϵ is the inverse image of Y in $\mathrm{Spec}(S_\epsilon)$, and $U_\epsilon = \mathrm{Spec}(S_\epsilon) \backslash Z_\epsilon$. Note that

$$S_\epsilon = (R \,\widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}})\langle u \rangle/(uf, p^\epsilon) = R_\epsilon[u]/(uf_\epsilon)$$

where $R_\epsilon = R \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon)$ and $f_\epsilon \in R_\epsilon$ is the image of $f \in R$.

Let $W \subset \mathrm{Spec}(S_\epsilon)$ be the preimage of $V = V(\overline{f}) \subset \mathrm{Spec}(\overline{R})$. Then $W = V \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathbb{A}^1_{\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon}$ is affine. The map $S_\epsilon \to R_\epsilon$ sending u to zero induces a section $\mathrm{Spec}(R_\epsilon) \to \mathrm{Spec}(S_\epsilon)$. We have a decomposition $\mathrm{Spec}(S_\epsilon) = N \cup W$, where $N = \mathrm{Spec}(R_\epsilon[u]/(u)) \simeq \mathrm{Spec}(R_\epsilon)$ is the image of the section $\mathrm{Spec}(R_\epsilon) \to \mathrm{Spec}(S_\epsilon)$. Take $V_\epsilon = V \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon)$. Then $W = V_\epsilon \times_{\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon)} \mathbb{A}^1_{\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon}$, and $N \cap W = V_\epsilon$.

We then have the following interpretations:

(1) Each section in $\Gamma(\mathrm{Spec}(S_\epsilon), \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$ is a pair (f_1, f_2) such that $f_1 \in \Gamma(N, \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$ and $f_2 \in \Gamma(W, \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$ such that $f_1 = f_2$ on $N \cap W = V_\epsilon$.

(2) Each section in $H^0(U_\epsilon, \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$ is a pair (f_1, f_2) such that $f_1 \in H^0(U_\epsilon \cap N, \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$, and $f_2 \in H^0(U_\epsilon \cap W, \mathcal{O}_{\mathrm{Spec}(S_\epsilon)})$, such that $f_1 = f_2$ on $U_\epsilon \cap N \cap W$.

The (classical) Hartog's extension principle, i.e. Lemma 2.2.4 applied to $Y \subset \mathrm{Spec}(\overline{R})$, shows that

$$\Gamma(\mathrm{Spec}(\overline{R}) \backslash Y) \simeq \Gamma(\mathrm{Spec}(\overline{R})).$$

Under base-change this gives

$$\Gamma(U_\epsilon \cap N) \simeq \Gamma(\mathrm{Spec}(\overline{Y}) \backslash Y) \otimes_{\mathbb{F}_p} \mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon \simeq \Gamma(\mathrm{Spec}(\overline{R})) \otimes_{\mathbb{F}_p} \mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon \simeq \Gamma(N).$$

Thus injectivity reduces to show that

$$\Gamma(V) \otimes_{\mathbb{F}_p} (\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon)[u] = \Gamma(W) \to \Gamma(U_\epsilon \cap W) = \Gamma(V \backslash Y) \otimes_{\mathbb{F}_p} (\mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon)[u]$$

is injective. It suffices to show that $\Gamma(V) \to \Gamma(V \backslash Y)$ is injective, where both V and $V \backslash Y$ are $\mathbb{F}_p$-schemes. We have $\mathrm{depth}(\mathcal{O}_{V,y}) = \mathrm{depth}(\overline{R}_y) - 1$ for all $y \in V$, cf. [Sta, Tag 090R]. Thus $\mathrm{depth}(\mathcal{O}_{V,y}) \geq 1$ for every $x \in V \cap Y$ by Serre's criterion, i.e. Lemma 2.2.3. Then the desired injectivity follows from Lemma 2.2.2.

It remains to prove the surjectivity. Let S' be the u-adic completion of S equipped with the (p, u)-adic topology. We have a natural injection $S \to S'$. Since $u \cdot f = p^\epsilon$, the topology of S' is also u-adic. Hence $|\mathrm{Spf}(S')| = |\mathrm{Spec}(S_\epsilon)|$ is a closed subspace of $|\mathrm{Spf}(S)|$. The first step is to prove the surjectivity of $S' \to H^0(U \cap |\mathrm{Spf}(S')|, \mathcal{O}_{\mathrm{Spf}(S')})$. By modulo u, it sufficies to show the surjectivity of

$$\overline{R} \otimes_{\mathbb{F}_p} \mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon = R_\epsilon \to H^0(U \cap \mathrm{Spec}(R_\epsilon), \mathcal{O}_{\mathrm{Spec}(R_\epsilon)}) = H^0(U \cap \mathrm{Spec}(\overline{R}), \mathcal{O}_{\mathrm{Spec}(\overline{R})}) \otimes_{\mathbb{F}_p} \mathbb{Z}_p^{\mathrm{cycl}}/p^\epsilon.$$

Lemma 2.2.4 shows that the map

$$\overline{R} \to H^0(\mathrm{Spec}(\overline{R})\backslash Y, \mathcal{O}_{\mathrm{Spec}(\overline{R})}) = H^0(\mathrm{Spec}(\overline{R}) \cap U, \mathcal{O}_{\mathrm{Spec}(R_\epsilon)})$$

is an isomorphism. From here the desired surjectivity is clear. □

2.3. **Tate's normalized traces.**

Lemma 2.3.1. Let R be a p-adically complete flat $\mathbb{Z}_p$-algebra. Let $Y_1, \ldots, Y_n \in R$. Let $P_1, \ldots, P_n \in R\langle X_1, \ldots, X_n\rangle$ be topologically nilpotent elements, or equivalently, each P_i has topologically nilpotent coefficients in R. Let

$$S = R\langle X_1, \ldots, X_n\rangle/(X_1^p - Y_1 - P_1, \ldots, X_n - Y_n - P_n).$$

Then

(1) The ring S is a finite free R-module of rank p^n, with a basis given by $X_1^{i_1} \cdots X_n^{i_n}$ with $0 \le i_1, \ldots, i_n \le p-1$.
(2) Let I be the ideal of R generated by p together with all the coefficients of all P_i. Then the trace map $\mathrm{tr}_{S/R} : S \to R$ sends S to I^n, i.e. $\mathrm{tr}_{S/R}(S) \subset I^n$.

Proof. Omitted. □

Lemma 2.3.2. Let R be a p-adically complete flat $\mathbb{Z}_p$-algebra topologically of finite type, formally smooth of dimension n over $\mathbb{Z}_p$. Let $f \in R$ such that its reduction $\overline{f} \in \overline{R} = R/p$ is not a zero-divisor. Let $0 \le \epsilon < 1/2$. Let

$$S_\epsilon = (R \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}})\langle u_\epsilon\rangle/(u_\epsilon \cdot f - p^\epsilon).$$

Suppose $\varphi : S_\epsilon \to S_{\epsilon/p}$ is a map of $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra such that modulo $p^{1-\epsilon}$ it is given by the relative Frobenius. In other words, $\varphi \mod p^{1-\epsilon}$ is the map

$$R_{1-\epsilon}[u_\epsilon]/(f \cdot u_\epsilon - p^\epsilon) \to R_{1-\epsilon}[u_{\epsilon/p}]/(f \cdot u_{\epsilon/p} - p^{\epsilon/p}),$$

where $R_{1-\epsilon} = \overline{R} \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p^{\mathrm{cycl}}/p^{1-\epsilon})$, which sends u_ϵ to $u_{\epsilon/p}^p$, and restricts to $\mathrm{Fr}_{\overline{R}} \otimes \mathrm{id}$ on $R_{1-\epsilon}$. Then

(1) The map
$$\varphi[1/p] : S_\epsilon[1/p] \to S_{\epsilon/p}[1/p]$$
is finite and flat of degree p^n.
(2) The trace map
$$\mathrm{tr} = \mathrm{tr}_{S_{\epsilon/p}[1/p]/S_\epsilon[1/p]} : S_{\epsilon/p}[1/p] \to S_\epsilon[1/p]$$
sends $S_{\epsilon/p}$ into $p^{n-(2n+1)\epsilon} S_\epsilon$. Here $S_{\epsilon/p}[1/p]$ is viewed as an $S_\epsilon[1/p]$-algebra via $\varphi[1/p]$.

Proof. Omitted. □

2.4. **Riemann's Hebbarkeitssatz.**

Definition 2.4.1. Let p be a prime. Let K be a perfectoid field (of any characteristic). Let t be a non-zero element of K with $|p| \le |t| < 1$. A triple $(\mathcal{X}, \mathcal{Z}, \mathcal{U})$, where $\mathcal{X}$ is an affinoid perfectoid space over K, $\mathcal{Z}$ is a closed subset of $\mathcal{X}$, and $\mathcal{U}$ is a quasi-compact open subset of $\mathcal{X}\backslash\mathcal{Z}$, is said to be good, if

$$H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^+/t)^a \simeq H^0(\mathcal{X}\backslash\mathcal{Z}, \mathcal{O}_{\mathcal{X}}^+/t)^a \hookrightarrow H^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}}^+/t)^a.$$

Remark 2.4.2. This notion is independent of the choice of t, and is compatible with tilting.

Situation 2.4.3. Let $K = \mathbb{F}_p((t^{1/p^\infty}))$. Let R_0 be a reduced Tate K-algebra topologically of finite type. Let $\mathcal{X}_0 = \mathrm{Spa}(R_0, R_0^\circ)$ be the associated affinoid adic space of finite type over K. Let R be the completed perfection of R_0, which is a p-finite perfectoid K-algebra. Let $\mathcal{X} = \mathrm{Spa}(R, R^+)$ with $R^+ = R^\circ$, the associated p-finite affinoid perfectoid space over K. Let I_0 be an ideal of R_0. Let $I = I_0 R \subset R$. Let $\mathcal{Z}_0 = V(I_0) \subset \mathcal{X}_0$. Let $\mathcal{Z} = V(I) \subset \mathcal{X}$. Let $\mathcal{U}_0$ be a quasi-compact open subset of $\mathcal{X}_0\backslash\mathcal{Z}_0$ with preimage $\mathcal{U} \subset \mathcal{X}\backslash\mathcal{Z}$.

Lemma 2.4.4. Assume Situation 2.4.3. Suppose $(\mathcal{X}, \mathcal{Z}, \mathcal{U})$ is good. Suppose that R_0 is normal, and that $V(I_0) \subset \mathrm{Spec}(R_0)$ is of codimension ≥ 2. Let R_0' be a finite normal R_0-algebra which is étale outside $V(I_0)$, and such that no irreducible component of $\mathrm{Spec}(R_0')$ maps into $V(I_0)$. Let $I_0' = I_0 R_0'$, and $\mathcal{U}_0' \subset \mathcal{X}_0'$ the preimage of $\mathcal{U}_0$. Let R', I', $\mathcal{X}'$, $\mathcal{Z}'$, $\mathcal{U}'$ be the associated perfectoid objects.

(1) There is a perfect trace pairing

$$\mathrm{tr}_{R_0'/R_0} : R_0' \otimes_{R_0} R_0' \to R_0.$$

(2) The trace pairing induces a trace pairing

$$\mathrm{tr}_{R'^\circ/R^\circ} : R'^\circ \otimes_{R^\circ} R'^\circ \to R^\circ.$$

which is almost perfect.

(3) For all open subsets $\mathcal{V} \subset \mathcal{X}$ with preimage $\mathcal{V}' \subset \mathcal{X}'$, the trace pairing induces an isomorphism

$$H^0(\mathcal{V}', \mathcal{O}^+_{\mathcal{X}'}/t)^a \simeq \mathrm{Hom}_{R^\circ/t}(R'^\circ/t, H^0(\mathcal{V}, \mathcal{O}^+_{\mathcal{X}}/t))^a.$$

(4) The triple $(\mathcal{X}', \mathcal{Z}', \mathcal{U}')$ is good.

(5) If $\mathcal{X}' \to \mathcal{X}$ is surjective, then the map

$$H^0(\mathcal{X}, \mathcal{O}^+_{\mathcal{X}}/t) \to H^0(\mathcal{X}', \mathcal{O}^+_{\mathcal{X}'}/t) \cap H^0(\mathcal{U}, \mathcal{O}^+_{\mathcal{X}}/t)$$

is an almost isomorphism.

Proof. Omitted. □

Lemma 2.4.5. Suppose we have a filtered inductive system $(R_0^{(i)})_{i\in I}$ as in the previous lemma, giving rise to $\mathcal{X}^{(i)}$, $\mathcal{Z}^{(i)}$, $\mathcal{U}^{(i)}$. Assume that all transition maps $\mathcal{X}^{(i)} \to \mathcal{X}^{(j)}$ are surjective. Let $\widetilde{\mathcal{X}}$ be the inverse limit of the $\mathcal{X}^{(i)}$ in the category of perfectoid spaces over K, with preimage $\widetilde{\mathcal{Z}} \subset \widetilde{\mathcal{X}}$ of $\mathcal{Z}$, and $\widetilde{\mathcal{U}} \subset \widetilde{\mathcal{X}} \backslash \widehat{\mathcal{U}}$ of $\mathcal{U}$. Then the triple $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{Z}}, \widetilde{\mathcal{U}})$ is good.

Proof. Omitted. □

Lemma 2.4.6. Assume Situation 2.4.3. Let A_0 be a ring that is normal, of finite type over $\mathbb{F}_p$, and admitting a resolution of singularities. Assume further that

(1) $R_0 = (A_0 \widehat{\otimes}_{\mathbb{F}_p} K)\langle u\rangle/(uf - t)$ for some non-zero-divisor $f \in A_0$.
(2) $I_0 = JR_0$ for some ideal $J \subset A_0$ with $V(J) \subset \mathrm{Spec}(A_0)$ of codimension ≥ 2.
(3) $\mathcal{U}_0 = \{x \in \mathcal{X}_0 \mid \exists g \in J, |g(x)| = 1\}$.

Then the triple $(\mathcal{X}, \mathcal{Z}, \mathcal{U})$ is good.

Proof. Omitted. □

2.5. **The Hodge–Tate filtration.**

Lemma 2.5.1. Let C be an algebraically closed and complete extension of $\mathbb{Q}_p$. Let $A \to \mathrm{Spec}(C)$ be an Abelian variety. Then A has its Hodge–Tate filtration

$$0 \to \mathrm{Lie}(A)(1) \to T_p(A) \otimes_{\mathbb{Z}_p} C \to (\mathrm{Lie}(A^\vee))^* \to 0.$$

((todo: ...))

3. Siegel Modular Varieties

Let p be a fixed prime.

Definition 3.0.1. The symplectic similitude group GSp_{2g} is the reductive group scheme over $\mathbb{Z}$ whose points in a commutative ring R are given by

$$\mathrm{GSp}_{2g}(R) = \{x \in \mathrm{GL}_{2g}(V); \exists \nu(x) \in R^\times, x^t\Omega x = \nu(x)\Omega\}$$

where $\Omega = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ is the standard symplectic matrix of order $2g$.

In the following discussion, we write $G = \mathrm{GSp}_{2g}$. Let $K_p = G(\mathbb{Z}_p)$. Let K^p be a compact open subgroup of $G(\mathbb{A}^{\infty,p})$ that is contained in

$$\Gamma(N)^{(p)} = \{g \in G(\mathbb{A}^{\infty,p}); g \equiv 1 \bmod N\}$$

for some integer $N \geq 3$ not divisible by p.

Definition 3.0.2. Let $m \geq 1$ be an integer.

$$\Gamma_0(p^m) = \left\{ g \in G(\mathbb{Z}_p); g \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \bmod p^m \right\}$$

$$\Gamma_{\mathrm{s}}(p^m) = \left\{ g \in G(\mathbb{Z}_p); g \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \bmod p^m, \nu(g) \equiv 1 \bmod p^m \right\}$$

$$\Gamma_1(p^m) = \left\{ g \in G(\mathbb{Z}_p); g \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \bmod p^m \right\}$$

$$\Gamma(p^m) = \left\{ g \in G(\mathbb{Z}_p); g \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bmod p^m \right\}$$

Definition 3.0.3. Let $\mathcal{S}$ be the Shimura datum associated to a symplectic vector space of dimension $2g$. Then we have the Shimura varieties $\mathrm{Sh}_K(\mathcal{S})$ for every compact open subgroup $K \subset \mathrm{GSp}_{2g}(\mathbb{A}_f)$.

Let X be the scheme over $\mathrm{Spec}(\mathbb{Z}_{(p)})$ classifying principally polarized projective Abelian schemes of relative dimension g with level K^p structures. Let X^* be the minimal compactification of X as constructed in [FC13, Chapter V].

For each $U \in \{\Gamma(p^m), \Gamma_{\mathrm{s}}(p^m), \Gamma_0(p^m)\}$, we set $X_{U,\mathbb{Q}} = \mathrm{Sh}_{K^pU}(\mathcal{S})$, which is a scheme over $\mathbb{Q}$ with certain moduli interpretations (see Remark 3.0.4).

Let $\mathfrak{X}$ be the formal scheme over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$ defined as the p-completion of $X_{\mathbb{Z}_p^{\mathrm{cycl}}} = X \times_{\mathrm{Spec}(\mathbb{Z}_{(p)})} \mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})$.

The universal Abelian scheme $A \to X$ gives a line bundle $\omega = \omega_{A/S} = \wedge^g \Omega^1_{A/X}$. The sheaf ω extends to the minimal compactification X^*. The Hasse invariant defines a section $\mathrm{Ha} \in H^0(X_{\mathbb{F}_p}, \omega^{\otimes(p-1)})$. The section Ha extends to $\mathrm{Ha} \in H^0(X^*_{\mathbb{F}_p}, \omega^{\otimes(p-1)})$.

Let $\mathfrak{A} \to \mathfrak{X}$ be the universal formal Abelian scheme.

Remark 3.0.4. The moduli interpretations can be described as follows.

(1) $\mathrm{Sh}_{K^pG(\mathbb{Z}_p),\mathbb{Z}_{(p)}}$ represents the following problem $S \mapsto \{(A, \lambda, \eta)\}/\sim$ where
- A is a projective Abelian scheme over S of relative dimension g.
- λ is a principal polarization of A.
- η is a level K^p structure on A.

(2) $\mathrm{Sh}_{K^p\Gamma(p^m),\mathbb{Q}}$ represents the following problem $S \mapsto \{(A, \lambda, \eta, \eta_p)\}/\sim$ where
- $(A, \lambda, \eta) \in \mathrm{Sh}_{K^pG(\mathbb{Z}_p),\mathbb{Z}_{(p)}}(S)$.
- η_p is a level $\Gamma(p^m)$ structure on A.

(3) $\mathrm{Sh}_{K^p\Gamma_0(p^m),\mathbb{Q}}$ represents the following problem $S \mapsto \{(A, \lambda, \eta, D)\}/\sim$ where
- $(A, \lambda, \eta) \in \mathrm{Sh}_{K^pG(\mathbb{Z}_p),\mathbb{Z}_{(p)}}(S)$.
- D is a totally isotropic subgroup of $A[p^m]$.

(4) $\mathrm{Sh}_{K^p\Gamma_{\mathrm{s}}(p^m),\mathbb{Q}}$ represents the following problem $S \mapsto \{(A, \lambda, \eta, D, t)\}/\sim$ where
- $(A, \lambda, \eta, D) \in \mathrm{Sh}_{K^p\Gamma_0(p^m),\mathbb{Q}}(S)$.
- $t : \mu_{p^m} \to \mathbb{Z}/p^m\mathbb{Z}$ is an isomorphism.

The first and second results are well-known, cf. [Kot92]. For the last two assertions, use the free action of $U/\Gamma(p^m)$ on the Shimura variety, where $U \in \{\Gamma_0(p^m), \Gamma_{\mathrm{s}}(p^m)\}$.

Definition 3.0.5. Let $\mathfrak{X}'$ be the formal scheme over $\mathrm{Spf}(\mathbb{F}_p[[t^{1/(p-1)p^\infty}]])$ given by the t-adic completion of $X \times_{\mathrm{Spec}(\mathbb{Z}_{(p)})} \mathrm{Spec}(\mathbb{F}_p[[t^{1/(p-1)p^\infty}]])$. Let $\mathcal{X}'$ be the generic fiber of the adic space associated to $\mathfrak{X}'$. Define $\mathfrak{X}'^*$ and $\mathfrak{A}^*$ similarly, with generic fibers $\mathcal{X}'^*$ and $\mathcal{A}'$.

Definition 3.0.6. Let $X^{\mathrm{ord}*} \subset X^* \times_{\mathrm{Spec}(\mathbb{Z}_{(p)})} \mathrm{Spec}(\mathbb{F}_p)$ be the locus where the Hasse invariant is invertible. Then $X^{\mathrm{ord}*}$ is affine over $\mathbb{F}_p$ (as it's cut out by an ample line bundle). Let $X^{\mathrm{ord}} \subset X \times_{\mathrm{Spec}(\mathbb{Z}_{(p)})} \mathrm{Spec}(\mathbb{F}_p)$ be the preimage of $X^{\mathrm{ord}*}$, which is the ordinary locus. Let D_m be the quotient $A^{\mathrm{ord}}[p^m]/C_m$, where $A^{\mathrm{ord}} \to X^{\mathrm{ord}}$ is the universal Abelian variety, and C_m is the canonical subgroup of level m. Let $X^{\mathrm{ord}}_{\Gamma_1(p^m)}$ be the scheme over X^{ord} parametrizing all the isomorphisms $D_m^{\mathrm{ord}} \simeq (\mathbb{Z}/p^m\mathbb{Z})^g$. Then $X^{\mathrm{ord}}_{\Gamma_1(p^m)} \to X^{\mathrm{ord}}$ is finite. Define

$$X^{\mathrm{ord}*}_{\Gamma_1(p^m)} = \mathrm{Spec}(H^0(X^{\mathrm{ord}}_{\Gamma_1(p^m)}, \mathcal{O}_{X^{\mathrm{ord}}_{\Gamma_1(p^m)}})).$$

Then map $X^{\mathrm{ord}*}_{\Gamma_1(p^m)} \to X^{\mathrm{ord}*}$ is a finite map of affine schemes over $\mathbb{F}_p$, such that $X^{\mathrm{ord}}_{\Gamma_1(p^m)}$ is the preimage of X^{ord}. Also $X^{\mathrm{ord}*}_{\Gamma_1(p^m)}$ is normal.

4. The anti-canonical towers

4.1. The Frobenius tower of formal models.

Lemma 4.1.1. Let S be a p-adically complete $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. There is a bijection

$$\mathrm{Hom}_{\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})}(\mathrm{Spf}(S), \mathfrak{X}^*) \simeq \mathrm{Hom}_{\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})}(\mathrm{Spec}(S), X^*_{\mathbb{Z}_p^{\mathrm{cycl}}}).$$

Speculation 4.1.2. ((todo: check: Let Y be a scheme over $\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})$. Let $\mathfrak{Y}$ be the formal scheme over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$ obtained as the p-completion of Y. Let S be a p-adically complete $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Then there is a bijection

$$\mathrm{Hom}_{\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})}(\mathrm{Spf}(S), \mathfrak{Y}) \simeq \mathrm{Hom}_{\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})}(\mathrm{Spec}(S), Y).$$

))

Definition 4.1.3. Let $\mathcal{N}_\epsilon$ be the functor sending a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra S to the set of pairs $(f, [u])$, where

- f is a map $\mathrm{Spf}(S) \to \mathfrak{X}^*$; it's equivalent to a map $\mathrm{Spec}(S) \to X^*_{\mathbb{Z}_p^{\mathrm{cycl}}}$ by Lemma 4.1.1.
- Let $\overline{f} : \mathrm{Spec}(S/p) \to X^*_{\mathbb{F}_p}$ be the reduction of $\mathrm{Spec}(S) \to X^*_{\mathbb{Z}_p^{\mathrm{cycl}}}$. Recall that we have the Hasse section $\mathrm{Ha} \in H^0(X^*_{\mathbb{F}_p}, \omega^{\otimes(p-1)})$. It pullbacks to $\overline{f}^*\mathrm{Ha} \in H^0(\mathrm{Spec}(S/p), \overline{f}^*\omega^{\otimes(p-1)})$. Then $[u]$ is an equivalence class of sections $u \in H^0(\mathrm{Spec}(S), f^*\omega^{\otimes(1-p)})$ satisfying $u \cdot \overline{f}^*\mathrm{Ha} = p^\epsilon \in S/p$ under the equivalence relation that $u \sim u'$ if and only if there exists some $h \in S$ such that $u' = u(1 + p^{1-\epsilon}h)$.

Lemma 4.1.4. Then the functor $\mathcal{N}_\epsilon$ is representable by a formal scheme flat over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$. For $\mathrm{Spf}(R) \subset \mathfrak{X}^*_{\mathbb{Z}_p}$, we have

$$\mathcal{N}_\epsilon \times_{\mathfrak{X}^*} \mathrm{Spf}(R \mathbin{\widehat{\otimes}_{\mathbb{Z}_p}} \mathbb{Z}_p^{\mathrm{cycl}}) = \mathrm{Spf}((R \mathbin{\widehat{\otimes}_{\mathbb{Z}_p}} \mathbb{Z}_p^{\mathrm{cycl}})\langle u\rangle/(u\widetilde{\mathrm{Ha}} - p^\epsilon))$$

where $\widetilde{\mathrm{Ha}} \in H^0(\mathrm{Spec}(R), \omega^{\otimes(p-1)})$ is a lift of $\mathrm{Ha} \in H^0(\mathrm{Spec}(R/p), \omega^{\otimes(p-1)})$.

Definition 4.1.5. Let $\mathfrak{X}(\epsilon) \to \mathfrak{X}$ be the pullback of $\mathfrak{X}^*(\epsilon) \to \mathfrak{X}^*$ along $\mathfrak{X} \to \mathfrak{X}^*$. Let $\mathfrak{A}(\epsilon) \to \mathfrak{X}(\epsilon)$ be the pullback of $\mathfrak{A} \to \mathfrak{X}$ along $\mathfrak{X}(\epsilon) \to \mathfrak{X}$.

Let $\mathcal{X}$ be the generic fiber of the adic space associated to the formal scheme $\mathfrak{X}$. Let $\mathcal{X}(\epsilon)$ be the generic fiber of the adic space associated to $\mathfrak{X}(\epsilon)$. Then $\mathcal{X}$ admits an open embedding to the X^{ad}, the adic space associated to the scheme $X_{\mathbb{Q}_p^{\mathrm{cycl}}}$. Let $\mathcal{X}_{\Gamma_s(p^m)}$ be the inverse image of $\mathcal{X}$ under the map $X^{\mathrm{ad}}_{\Gamma_s(p^m)} \to X^{\mathrm{ad}}$.

Remark 4.1.6. ((todo: moduli interpretation of $\mathfrak{X}(\epsilon)$. Should be almost identical to $\mathcal{M}_\epsilon$.))

Definition 4.1.7. For a formal scheme $\mathfrak{Y}$ over $\mathbb{Z}_p^{\mathrm{cycl}}$ and $a \in \mathbb{Z}_p^{\mathrm{cycl}}$, we write $\mathfrak{Y}/a$ for $\mathfrak{Y} \times_{\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})} \mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/a)$.

Definition 4.1.8. For a formal scheme $\mathfrak{Y}$ over $\mathbb{Z}_p^{\mathrm{cycl}}/p$, we write $\mathfrak{Y}^{(p)}$ for the pullback of $\mathfrak{Y}$ along the (absolute) Frobenius $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p) \to \mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p)$.

Lemma 4.1.9. We have a natural isomorphism

$$(\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)} \simeq \mathfrak{X}^*(\epsilon)/p$$

of formal schemes over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p)$. Furthermore, by pullback we get the following commutative diagram

$$\begin{array}{ccccc}
(\mathfrak{A}(p^{-1}\epsilon)/p)^{(p)} & \longrightarrow & (\mathfrak{X}(p^{-1}\epsilon)/p)^{(p)} & \longrightarrow & (\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)} \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{A}(\epsilon)/p & \longrightarrow & \mathfrak{X}(\epsilon)/p & \longrightarrow & \mathfrak{X}^*(\epsilon)/p
\end{array}$$

where each vertical map is an isomorphism.

Proof. Let S be a ((discrete? flat)) $(\mathbb{Z}_p^{\mathrm{cycl}}/p)$-algebra. Then

$$(\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)}(S) = (\mathfrak{X}^*(p^{-1}\epsilon)/p)(\mathrm{Fr}_*S),$$

where Fr_*S is the the $(\mathbb{Z}_p^{\mathrm{cycl}}/p)$-algebra obtained from S by precomposing with $\mathrm{Fr} : \mathbb{Z}_p^{\mathrm{cycl}}/p \to \mathbb{Z}_p^{\mathrm{cycl}}/p$. Each map $\mathrm{Spf}(\mathrm{Fr}_*S) \to \mathfrak{X}^*(p^{-1}\epsilon)/p$ is equivalent to a pair $(f, [u])$, where

- $f : \mathrm{Spec}(\mathrm{Fr}_*S) \to X^*_{\mathbb{Z}_p^{\mathrm{cycl}}}$ is a map over $\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})$.
- $u \in H^0(\mathrm{Spec}(\mathrm{Fr}_*S), f^*\omega^{\otimes(1-p)})$ is a section such that $u \cdot f^*\mathrm{Ha} = p^{p^{-1}\epsilon} \in \mathrm{Fr}_*S$. Note that $(\mathrm{Fr}_*S)/p = \mathrm{Fr}_*S$ since S is defined over $\mathbb{Z}_p^{\mathrm{cycl}}/p$.

Recall that $X^*_{\mathbb{Z}_p^{\mathrm{cycl}}} = X^*_{\mathbb{Z}_p} \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})$, and thus $(f, [u])$ is equivalent ((todo: should be more precise)) to the following datum

- $f : \mathrm{Spec}(\mathrm{Fr}_*S) \to X^*_{\mathbb{Z}_p}$ is a map over $\mathrm{Spec}(\mathbb{Z}_p)$.
- ((todo: Check the reduction of u)) $u \in H^0(\mathrm{Spec}(\mathrm{Fr}_*S), f^*\omega^{\otimes(1-p)})$ is a section such that $u \cdot f^*\mathrm{Ha} = p^{p^{-1}\epsilon} \in \mathrm{Fr}_*S$.

Note that the Frobenius on $\mathbb{Z}_p/p = \mathbb{F}_p$ is simply the identity, and thus the map $\mathrm{Spec}(\mathrm{Fr}_*S) \to \mathrm{Spec}(\mathbb{Z}_p)$ is identical to $\mathrm{Spec}(S) \to \mathrm{Spec}(\mathbb{Z}_p)$. But under this identification the element $p^{p^{-1}\epsilon} \in \mathrm{Fr}_*S$ corresponds to $p^\epsilon \in S$. Then $f : \mathrm{Spec}(\mathrm{Fr}_*S) \to X^*_{\mathbb{Z}_p}$ can be reinterpreted as a map $g : \mathrm{Spec}(S) \to X^*_{\mathbb{Z}_p}$ over $\mathrm{Spec}(\mathbb{Z}_p)$. We write $v = u$ for clarity. The section v then satisfies $v \cdot g^*\mathrm{Ha} = p^\epsilon \in S$. The pair $(g, [v])$ then corresponds to a map $\mathrm{Spf}(S) \to \mathfrak{X}^*(\epsilon)/p$ over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p)$. □

Lemma 4.1.10. The Frobenius map $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p) \to \mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p)$ induces the following commutative diagram

$$\begin{array}{ccccc}
\mathfrak{A}(p^{-1}\epsilon)/p & \longrightarrow & \mathfrak{X}(p^{-1}\epsilon)/p & \longrightarrow & \mathfrak{X}^*(p^{-1}\epsilon)/p \\
\downarrow & & \downarrow & & \downarrow \\
(\mathfrak{A}(p^{-1}\epsilon)/p)^{(p)} & \longrightarrow & (\mathfrak{X}(p^{-1}\epsilon)/p)^{(p)} & \longrightarrow & (\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)}
\end{array}$$

Proof. This follows from the universal property of pullback. □

Remark 4.1.11. ((todo: Explain the moduli interpretation of

$$\mathfrak{X}^*(p^{-1}\epsilon)/p \to (\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)} \simeq \mathfrak{X}^*(\epsilon)/p.$$

))

Speculation 4.1.12. ((todo: check: Let S be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let $f : \mathrm{Spf}(S) \to \mathfrak{X}$ be a map over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$. Let $A \to \mathrm{Spec}(S)$ be the corresponding Abelian scheme. Suppose $A \to \mathrm{Spec}(S)$ satisfies strong $O(1, \epsilon)$. Let C be the strong canonical subgroup of $A \to \mathrm{Spec}(S)$ of level 1. Then $B = A/C$ satisfies weak $O(1, \epsilon)$.))

Speculation 4.1.13. ((todo: cf. [Wed99]))

Lemma 4.1.14. There is a unique commutative diagram

$$\begin{array}{ccccc}
\mathfrak{A}(p^{-1}\epsilon) & \longrightarrow & \mathfrak{X}(p^{-1}\epsilon) & \longrightarrow & \mathfrak{X}^*(p^{-1}\epsilon) \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{A}(\epsilon) & \longrightarrow & \mathfrak{X}(\epsilon) & \longrightarrow & \mathfrak{X}^*(\epsilon)
\end{array}$$

that is identified with the following commutative diagram from Lemma 4.1.9 and Lemma 4.1.10, after modulo $p^{1-\epsilon}$.

$$\begin{array}{ccccc}
\mathfrak{A}(p^{-1}\epsilon)/p & \longrightarrow & \mathfrak{X}(p^{-1}\epsilon)/p & \longrightarrow & \mathfrak{X}^*(p^{-1}\epsilon)/p \\
\downarrow & & \downarrow & & \downarrow \\
(\mathfrak{A}(p^{-1}\epsilon)/p)^{(p)} & \longrightarrow & (\mathfrak{X}(p^{-1}\epsilon)/p)^{(p)} & \longrightarrow & (\mathfrak{X}^*(p^{-1}\epsilon)/p)^{(p)} \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{A}(\epsilon)/p & \longrightarrow & \mathfrak{X}(\epsilon)/p & \longrightarrow & \mathfrak{X}^*(\epsilon)/p
\end{array}$$

Proof. ((todo: finish the proof: The map $\mathfrak{X}(p^{-1}\epsilon) \to \mathfrak{X}(\epsilon)$ comes from the moduli interpretation, the weak canonical subgroup, and the Hasse invariant. Then $\mathfrak{A}(p^{-1}\epsilon) \to \mathfrak{A}(\epsilon)$ is obtained by base-change. The extension to $\mathfrak{X}^*(p^{-1}\epsilon) \to \mathfrak{X}^*(\epsilon)$ is done using Hartog's extension principle.))

We first construct the map $\mathfrak{X}(p^{-1}\epsilon) \to \mathfrak{X}(\epsilon)$. Let S be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let $(f, [u])$ be a pair where

- $f : \mathrm{Spf}(S) \to \mathfrak{X}$ is a map of formal schemes over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$; its equivalent to a map $f : \mathrm{Spec}(S) \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$.
- $u \in H^0(\mathrm{Spec}(S), f^*\omega^{\otimes(1-p)})$ is a section such that $u \cdot \overline{f}^*\mathrm{Ha} = p^{p^{-1}\epsilon} \in S/p$.

The map $f : \mathrm{Spec}(S) \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$ gives an Abelian scheme $A \to \mathrm{Spec}(S)$ ((todo: with principal polarization and level K^p structure)). We claim that $A \to \mathrm{Spec}(S)$ satisfies strong $O(1,\epsilon)$, i.e. $\mathrm{Ha}(A_1/\mathrm{Spec}(S_1))^p$ divides p^ϵ. This follows from

$$p^{p^{-1}\epsilon} = u \cdot \overline{f}^*\mathrm{Ha} = u \cdot \mathrm{Ha}(A_1/\mathrm{Spec}(S_1)).$$

Let $C \subset A[p]$ be the strong canonical subgroup of level 1. We get an Abelian scheme $A/C \to \mathrm{Spec}(S)$ ((todo: explain: equipped with induced polarization and level structure: use totally isotropic)), which corresponds to a map $g : \mathrm{Spec}(S) \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$. This gives a map $\mathfrak{X}(p^{-1}\epsilon) \to \mathfrak{X}$. We will show next that it can be factored as $\mathfrak{X}(p^{-1}\epsilon) \to \mathfrak{X}(\epsilon) \to \mathfrak{X}$.

((seems wrong: Then we declare that the pair $(f, [u])$ gets mapped to the pair $(g, [u^p])$.))

((seems wrong: By Speculation 4.1.12, the quotient $A/C \to \mathrm{Spec}(S)$ satisfies weak $O(1,\epsilon)$, i.e. there exists a section $v \in H^0(\mathrm{Spec}(S), \overline{g}^*\omega^{\otimes(1-p)})$ such that $v \cdot \overline{g}^*\mathrm{Ha} = p^\epsilon$. Then we declare that the pair $(f, [u])$ gets mapped to the pair $(g, [v])$. We need to check that $[v]$ is well-defined. ((wrong!)) It suffices to show that $\overline{g}^*\mathrm{Ha} = \mathrm{Ha}((A/C)_1/S_1)$ is not a zero-divisor. Otherwise, for every geometric point x of $\mathrm{Spec}(S)$, the Abelian scheme $(A/C)_x$ is not ordinary. This contradicts Speculation 4.1.13. Therefore we obtain a well-defined map $\mathfrak{X}(p^{-1}\epsilon) \to \mathfrak{X}(\epsilon)$.))

Let $B = A/C$. We have

$$p^\epsilon = u^p \cdot \mathrm{Ha}(A_1/\mathrm{Spec}(S_1))^p = u^p \cdot \mathrm{Ha}(A_1^{(p)}/\mathrm{Spec}(S_1)).$$

Modulo $p^{1-\epsilon}$,

$$p^\epsilon = u^p \cdot \mathrm{Ha}(A_{1-\epsilon}^{(p)}/\mathrm{Spec}(S_{1-\epsilon})) = u^p \cdot \mathrm{Ha}(B_{1-\epsilon}/\mathrm{Spec}(S_{1-\epsilon})).$$

Thus there is $v \in H^0(\mathrm{Spec}(S), g^*\omega^{\otimes(1-p)})$ such that $v = u^p \bmod p^{1-\epsilon}$ and $v \cdot \mathrm{Ha}(B_1/\mathrm{Spec}(S_1)) = p^\epsilon \bmod p^{1-\epsilon}$. Hence

$$v \cdot \mathrm{Ha}(B_1/\mathrm{Spec}(S_1)) = p^\epsilon + p^{1-\epsilon}t = p^\epsilon(1 + p^{1-2\epsilon}t) \in S/p$$

for some $t \in S$.

((check: $1 + p^{1-2\epsilon}t$ is invertible in S)) Then

$$(1 + p^{1-2\epsilon}t)^{-1}v \cdot \mathrm{Ha}(B_1/\mathrm{Spec}(S_1)) = p^\epsilon \in S/p.$$

(This shows that B is weak $O(1,\epsilon)$.) We claim that the pair $(f, [u])$ gets mapped to the pair $(g, [(1 + p^{1-2\epsilon}t)^{-1}v])$.

- First check this map is well-defined.
  - Any choice $u' \in [u]$ leads to $u^p = (u')^p \bmod p^{1-\epsilon}$.
  - Now choose another lift $v + p^{1-\epsilon}v'$ of v.
-

Another attempt at constructing the factorization $\mathfrak{X}(p^{-1}\epsilon) \to \mathfrak{X}(\epsilon) \to \mathfrak{X}$.

- We already know that B is weak $O(1,\epsilon)$, i.e. there exists a section $v \in H^0(\mathrm{Spec}(S/p), \overline{g}^*\omega^{\otimes(1-p)})$ such that
$$v \cdot \mathrm{Ha}(B_1/S_1) = p^\epsilon \bmod p.$$
- Modulo $p^{1-\epsilon}$,
$$p^\epsilon = v \cdot \mathrm{Ha}(B_{1-\epsilon}/S_{1-\epsilon}) = v \cdot \mathrm{Ha}(A_{1-\epsilon}^{(p)}/S_{1-\epsilon}) = u^p \cdot \mathrm{Ha}(A_{1-\epsilon}^{(p)}/S_{1-\epsilon}) \bmod p^{1-\epsilon}.$$
- Then
$$(v - u^p) \cdot \mathrm{Ha}(A_{1-\epsilon}^{(p)}/S_{1-\epsilon}) = 0 \bmod p^{1-\epsilon}.$$
- Maybe $\mathrm{Ha}(A_{1-\epsilon}^{(p)}/S_{1-\epsilon})$ is not a zero-divisor in $S/p^{1-\epsilon}$.

- Then $v = u^p \bmod p^{1-\epsilon}$.

Another attempt at constructing the factorization $\mathfrak{X}(p^{-1}\epsilon) \to \mathfrak{X}(\epsilon) \to \mathfrak{X}$.

- Let $\mathrm{Spf}(R) \subset \mathfrak{X}$ on which ω is trivial.
- Choose a lift $\widetilde{\mathrm{Ha}} \in H^0(\mathrm{Spec}(R), \omega^{\otimes(p-1)})$ of $\mathrm{Ha} \in H^0(\mathrm{Spec}(R/p), \omega^{\otimes(p-1)})$.
- We want

$$\begin{array}{ccc} & & \mathrm{Spf}(R\langle u\rangle/(u\cdot\widetilde{\mathrm{Ha}} - p^{\epsilon})) \\ & \nearrow & \downarrow \\ \mathrm{Spf}(R\langle u\rangle/(u\cdot\widetilde{\mathrm{Ha}} - p^{p^{-1}\epsilon})) & \longrightarrow & \mathrm{Spf}(R). \end{array}$$

  In other words,

$$\begin{array}{ccc} & & R\langle u\rangle/(u\cdot\widetilde{\mathrm{Ha}} - p^{\epsilon}) \\ & \swarrow & \uparrow \\ R\langle u\rangle/(u\cdot\widetilde{\mathrm{Ha}} - p^{p^{-1}\epsilon}) & \longleftarrow & R. \end{array}$$

We need to show that $\mathfrak{X}(p^{-1}\epsilon) \to \mathfrak{X}(\epsilon)$ extends to $\mathfrak{X}^*(p^{-1}\epsilon) \to \mathfrak{X}^*(\epsilon)$ ((cf. the remark in Lemma 4.1.4)).

- We'd like to apply Lemma 2.2.6 for the case $g \geq 2$.
- Let $\mathrm{Spf}(R) \subset \mathfrak{X}^*_{\mathbb{Z}_p}$ ((such that $\omega^{\otimes(p-1)}$ is trivial on $\mathrm{Spf}(R)$)). This gives an affine open $\mathrm{Spf}(R \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}})$ of $\mathfrak{X}^*$, and such affines cover $\mathfrak{X}^*$.
- Check that R is a topologically finitely generated flat p-adically complete $\mathbb{Z}_p$-algebra, and that R/p is normal.
- Check that $\mathrm{Ha} \in H^0(\mathrm{Spec}(R/p), \omega^{\otimes(p-1)}) \simeq R/p$ not a zero-divisor, where ω is the natural (ample) line bundle on $X^*_{\mathbb{F}_p}$.
  - $\mathrm{Spec}(R/p)$ is an affine open in $X^*_{\mathbb{F}_p}$ as $\mathrm{Spec}(R)$ is an affine open of $X^*_{\mathbb{Z}_p}$.
  - We have inclusion of opens
$$X^{\mathrm{ord}}_{\mathbb{F}_p} \subset X_{\mathbb{F}_p} \subset X^*_{\mathbb{F}_p}.$$
    The first inclusion is dense by Lemma **??**, and the second is dense by the property of minimal compactification ((todo: add reference)).
  - Thus the intersection
$$\mathrm{Spec}(R/p) \cap X^{\mathrm{ord}}_{\mathbb{F}_p}$$
    is non-empty.
  - Therefore Ha is not a zero-divisor since it is non-zero at a point.
- We need a map
$$\mathfrak{X}^*(p^{-1}\epsilon) \times_{\mathfrak{X}^*} \mathrm{Spf}(R \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}}) \to \mathfrak{X}^*(\epsilon) \times_{\mathfrak{X}^*} \mathrm{Spf}(R \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}})$$
- Choose a lift $\widetilde{\mathrm{Ha}} \in H^0(\mathrm{Spec}(R), \omega^{\otimes(p-1)}) \simeq R$ of $\mathrm{Ha} \in R/p$.
- Let $S_\epsilon = (R \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}})\langle u\rangle/(u\cdot\widetilde{\mathrm{Ha}} - p^{\epsilon})$. Let $S_{p^{-1}\epsilon} = (R \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}})\langle u\rangle/(u\cdot\widetilde{\mathrm{Ha}} - p^{p^{-1}\epsilon})$. Then we need a map
$$\begin{array}{ccc} \mathrm{Spf}(S_{p^{-1}\epsilon}) & \dashrightarrow & \mathrm{Spf}(S_\epsilon) \\ \downarrow & & \downarrow \\ \mathfrak{X}^*(p^{-1}\epsilon) & \dashrightarrow & \mathfrak{X}^*(\epsilon) \end{array}$$
- Consider the pullback diagram
$$\begin{array}{ccc} U & \longrightarrow & \mathrm{Spec}(R/p) \\ \downarrow & & \downarrow \\ X_{\mathbb{F}_p} & \longrightarrow & X^*_{\mathbb{F}_p}. \end{array}$$
  Then U is an open in $\mathrm{Spec}(R/p)$. Let Y be the complement of U in $\mathrm{Spec}(R/p)$.

- Check that Y has codimension ≥ 2 in $\mathrm{Spec}(R/p)$. This follows from that $Y \cap X_{\mathbb{F}_p} = \emptyset$ and that boundary of $X^*_{\mathbb{F}_p}$ has codimension $g \geq 2$.
- Let Z be the preimage of Y in $\mathrm{Spf}(S_\epsilon)$. Then Lemma 2.2.6 shows that the natural map
$$H^0(\mathrm{Spf}(S_\epsilon), \mathcal{O}_{\mathrm{Spf}(S_\epsilon)}) \to H^0(\mathrm{Spf}(S_\epsilon)\backslash Z, \mathcal{O}_{\mathrm{Spf}(S_\epsilon)})$$
is an isomorphism.
- Define $\mathfrak{U}$ by the pullback diagram
$$\begin{array}{ccc} \mathfrak{U} & \longrightarrow & \mathrm{Spf}(R \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}}) \\ \downarrow & & \downarrow \\ \mathfrak{X} & \longrightarrow & \mathfrak{X}^*. \end{array}$$
- Now we need to construct
$$\begin{array}{ccc} \mathfrak{X}(p^{-1}\epsilon) \times_{\mathfrak{X}} \mathfrak{U} & \longrightarrow & \mathrm{Spf}(S_{p^{-1}\epsilon}) \\ \downarrow & & \vdots \\ \mathfrak{X}(\epsilon) \times_{\mathfrak{X}} \mathfrak{U} & \longrightarrow & \mathrm{Spf}(S_\epsilon). \end{array}$$
- We claim that $\mathfrak{X}(\epsilon) \times_{\mathfrak{X}} \mathfrak{U} = \mathrm{Spf}(S_\epsilon)\backslash Z$.

□

4.2. The anti-canonical tower of level Γ_{s}.

Construction 4.2.1. Let $m \geq 1$.

We first construct a map $\mathfrak{X}(p^{-m}\epsilon) \to \mathfrak{X}$. Let S be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Let $\mathrm{Spf}(S) \to \mathfrak{X}(p^{-m}\epsilon)$ be a map over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$. It corresponds to a pair $(f, [u])$ where

- $f : \mathrm{Spec}(S) \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$ is a map over $\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{cycl}})$.
- $u \in H^0(\mathrm{Spec}(S), f^*\omega^{\otimes(1-p)})$ is a section such that $u \cdot \overline{f}^* \mathrm{Ha} = p^{p^{-m}\epsilon}$ in S/p.

The map $f : \mathrm{Spec}(S) \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$ gives an Abelian scheme $A \to \mathrm{Spec}(S)$. The section u shows that $A \to \mathrm{Spec}(S)$ satisifies strong $O(m, \epsilon)$, and thus has a strong canonical subgroup $C_m \subset A[p^m]$ of level m. The Abelian scheme $A/C_m \to \mathrm{Spec}(S)$ has induced principal polarization and level structure, and thus corresponds to a map $\mathrm{Spec}(S) \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$, which gives a map $\mathrm{Spf}(S) \to \mathfrak{X}$ over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$.

Passing to the adic fiber (i.e. the generic fiber of the associated adic space), we get a map $\mathcal{X}(p^{-m}\epsilon) \to \mathcal{X}$ of adic spaces. Now we construct a factorization $\mathcal{X}(p^{-m}\epsilon) \to \mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)} \to \mathcal{X}$, where the map $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)} \to \mathcal{X}$ is given by the moduli interpretation "$(A, D) \mapsto A/D$".

((todo: construct the factorization))

Lemma 4.2.2. For each $m \geq 1$, the $\mathcal{X}(p^{-m}\epsilon) \to \mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}$ extends uniquely to $\mathcal{X}^*(p^{-m}\epsilon) \to \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}$, and both maps are open immersions of adic spaces. Moreover, the following diagram
$$\begin{array}{ccc} \mathcal{X}^*(p^{-m-1}\epsilon) & \longrightarrow & \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^{m+1})} \\ \downarrow & & \downarrow \\ \mathcal{X}^*(p^{-m}\epsilon) & \longrightarrow & \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)} \end{array}$$
is a pullback diagram for all $m \geq 1$, where the vertical map on the left is induced from the map $\mathfrak{X}(p^{-m-1}\epsilon) \to \mathfrak{X}(p^{-m}\epsilon)$, cf. Lemma 4.1.14.

Proof. ((todo: write down the proof))

- Extension to minimal compactification:
- Open immersion:
  - The map $\theta : X_{\mathbb{Q}_p^{\mathrm{cycl}}} \to X_{\mathbb{Q}_p^{\mathrm{cycl}}}$ defined by $A \mapsto A/A[p^m]$ is an isomorphism.

  - The following diagram

$$\begin{array}{ccc} \mathcal{X}(p^{-m}\epsilon) & \longrightarrow & \mathcal{X} \\ \big\downarrow & & \big\downarrow \\ X^{\mathrm{ad}} & \xrightarrow{\theta} & X^{\mathrm{ad}} \end{array}$$

    commutes.
  - So the composition $\mathcal{X}^*(p^{-m}\epsilon) \to \mathcal{X}_{\Gamma_\mathrm{s}(p^m)} \to \mathcal{X}$ is an open immersion.
  - The map $\mathcal{X}_{\Gamma_\mathrm{s}(p^m)} \to \mathcal{X}$ is finite étale.
  - Thus the map $\mathcal{X}(p^{-m}\epsilon) \to \mathcal{X}_{\Gamma_\mathrm{s}(p^m)}$ is an open immersion.
  - Then pass to minimal compactification as follows.
  - We'd like to apply Lemma 2.2.6.
- Pullback diagram:
  - First show that

$$\begin{array}{ccc} \mathcal{X}(p^{-m-1}\epsilon) & \longrightarrow & \mathcal{X}_{\Gamma_\mathrm{s}(p^{m+1})} \\ \big\downarrow & & \big\downarrow \\ \mathcal{X}(p^{-m}\epsilon) & \longrightarrow & \mathcal{X}_{\Gamma_\mathrm{s}(p^m)} \end{array}$$

    is pullback diagram.
  - Commutativity of the diagram:
  - It is a pullback since both vertical maps are finite étale of degree $p^{g(g+1)/2}$.
  - Then pass to minimal compactification.

□

Definition 4.2.3. Let $\mathcal{X}_{\Gamma_\mathrm{s}(p)}(\epsilon)$ be the pullback of $\mathcal{X}(\epsilon)$ along $\mathcal{X}_{\Gamma_\mathrm{s}(p)} \to \mathcal{X}$.

Lemma 4.2.4. The following diagram

$$\begin{array}{ccc} \mathcal{X}(p^{-1}\epsilon) & \longrightarrow & \mathcal{X}_{\Gamma_\mathrm{s}(p)}(\epsilon) \\ \big\downarrow & & \big\downarrow \\ \mathcal{X}(\epsilon) & \xrightarrow{\mathrm{id}} & \mathcal{X}(\epsilon) \end{array}$$

commutes. Moreover, the map $\mathcal{X}(p^{-1}\epsilon) \to \mathcal{X}_{\Gamma_\mathrm{s}(p)}(\epsilon)$ is an open immersion, and the image of $\mathcal{X}(p^{-1}\epsilon)$ in $\mathcal{X}_{\Gamma_\mathrm{s}(p)}(\epsilon)$ is both open and closed.

Definition 4.2.5. ((todo: how to make this statement precise? do we actually need this?: Let $\mathcal{X}_{\Gamma_s(p)}(\epsilon)_a$ be the open and closed subset of $\mathcal{X}_{\Gamma_s(p)}(\epsilon)$ "parametrizing those $D \subset \mathcal{A}(\epsilon)[p]$ with $D \cap C = \{0\}$".))

Let $\mathcal{X}_{\Gamma_s(p)}(\epsilon)_a$ be the image of $\mathcal{X}(p^{-1}\epsilon)$ in $\mathcal{X}_{\Gamma_\mathrm{s}(p)}(\epsilon)$. Let $\mathcal{X}^*_{\Gamma_s(p)}(\epsilon)_a$ be the image of $\mathcal{X}^*(p^{-1}\epsilon)$ in $\mathcal{X}^*_{\Gamma_\mathrm{s}(p)}(\epsilon)$. Let $\mathcal{X}^*_{\Gamma_\mathrm{s}(p^m)}(\epsilon)_a$ be the pullback of $\mathcal{X}^*_{\Gamma_s(p)}(\epsilon)_a$ along $\mathcal{X}^*_{\Gamma_\mathrm{s}(p^m)}(\epsilon) \to \mathcal{X}^*_{\Gamma_\mathrm{s}(p)}(\epsilon)$.

Remark 4.2.6. $\mathcal{X}^*_{\Gamma_s(p)}(\epsilon)_a$ is both open and closed in $\mathcal{X}^*_{\Gamma_\mathrm{s}(p)}(\epsilon)$.

Lemma 4.2.7. For m sufficiently large, $\mathcal{X}^*_{\Gamma_\mathrm{s}(p^m)}(\epsilon)_a$ is affinoid.

Proof. ((todo: write down the proof))

- There exists an integer $m \geq 0$ such that $H^i(X^*_{\mathbb{Z}_p}, \omega^{\otimes p^m(p-1)}) = 0$ for all $i \geq 1$, since ω is an ample line bundle on $X^*_{\mathbb{Z}_p}$.
- We can find a lift $s \in H^0(X^*_{\mathbb{Z}_p}, \omega^{\otimes p^m(p-1)})$ lifting $\mathrm{Ha}^{p^m} \in H^0(X^*_{\mathbb{F}_p}, \omega^{\otimes p^m(p-1)})$. ((todo: add a proof; should follows from vanishing of first cohomology; maybe use a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules, and then pass to a long exact sequence))
- The condition $|\mathrm{Ha}| \geq |p|^{p^{-m}\epsilon}$ is equivalent to $|s| \geq |p|^\epsilon$.
- The condition defines an affinoid space $\mathcal{X}^*(p^{-m}\epsilon) \simeq \mathcal{X}^*_{\Gamma_\mathrm{s}(p^m)}(\epsilon)_a$.

□

Lemma 4.2.8. There exists a unique perfectoid space $\mathcal{X}^*_{\Gamma_s(p^\infty)}(\epsilon)_a$ such that

$$\mathcal{X}^*_{\Gamma_s(p^\infty)}(\epsilon)_a \sim \lim_m \mathcal{X}^*_{\Gamma_s(p^m)}(\epsilon)_a.$$

Similar results hold for $\mathcal{X}_{\Gamma_s(p^\infty)}(\epsilon)_a$ and $\mathcal{A}_{\Gamma_s(p^\infty)}(\epsilon)_a$.

Proof. ((todo: use tilting))

- Define
$$\mathfrak{X}^*_{\Gamma_s(p^\infty)}(\epsilon)_a = \lim_m \mathfrak{X}(p^{-m}\epsilon),$$
where the inverse limit is taken in the category of formal schemes over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$. Note that the transition maps are finite.
- Let $\mathrm{Spf}(R_{m_0}) \subset \mathfrak{X}(p^{-m_0}\epsilon)$ be affine. Let $\mathrm{Spf}(R_m) \subset \mathfrak{X}(p^{-m}\epsilon)$ be the preimage of $\mathrm{Spf}(R_{m_0})$ for $m \geq m_0$.
- We get an affine open $\mathrm{Spf}(R_\infty)$ of $\mathfrak{X}^*_{\Gamma_s(p^\infty)}(\epsilon)_a$, where R_∞ is the p-adic completion of $\mathrm{colim}_m R_m$. Then R_∞ is flat over $\mathbb{Z}_p^{\mathrm{cycl}}$.
- The transition map $R_m/p^{1-\epsilon} \to R_{m+1}/p^{1-\epsilon}$ agrees with the relative Frobenius. The absolute Frobenius then induces an isomorhpism
$$R_\infty/p^{(1-\epsilon)/p} = \mathrm{colim}_m R_{m+1}/p^{(1-p)/p} \simeq \mathrm{colim}_m R_m/p^{1-\epsilon} = R_\infty/p^{1-\epsilon}.$$
- Thus R_∞^a is a perfectoid $\mathbb{Z}_p^{\mathrm{cycl},a}$-algebra, cf. [Sch12, Definition 5.1.(ii)].
- Then $R_\infty[1/p]$ is a perfectoid $\mathbb{Q}_p^{\mathrm{cycl}}$-algebra, cf. [Sch12, Lemma 5.6].
- Then the generic fiber of $\mathfrak{X}^*_{\Gamma_s(p^\infty)}(\epsilon)_a$ is a perfectoid space $\mathcal{X}^*_{\Gamma_s(p^\infty)}(\epsilon)_a$ over $\mathbb{Q}_p^{\mathrm{cycl}}$, and
$$\mathcal{X}^*_{\Gamma_s(p^\infty)}(\epsilon)_a \sim \lim_m \mathcal{X}^*_{\Gamma_s(p^m)}(\epsilon)_a,$$
cf. [SW13, Definition 2.4.1, Proposition 2.4.2].
- Uniqueness follows from [SW13, Proposition 2.4.5].

□

Lemma 4.2.9. The tilt $\mathcal{X}^*_{\Gamma_s(p^\infty)}(\epsilon)_a^\flat$ identifies naturally with the open subset $\mathcal{X}'^{*\mathrm{perf}}(\epsilon) \subset \mathcal{X}'^{*\mathrm{perf}}$ where $|\mathrm{Ha}| \geq |t|^\epsilon$. The similar result holds for $\mathcal{A}$.

Proof. ((todo: split the proof))

- We define $\mathfrak{X}'^*(\epsilon) \to \mathfrak{X}'^*$ in a way similar to $\mathfrak{X}^*(\epsilon) \to \mathfrak{X}^*$, parametrizing sections $u \in \omega^{\otimes(1-p)}$ such that $u \cdot \mathrm{Ha} = t^\epsilon$.
- We have the map
$$\mathfrak{X}'^*(p^{-1}\epsilon) \to \mathfrak{X}'^*(\epsilon)$$
given by the raltive Frobenius.
- The inverse limit $\lim_m \mathfrak{X}'^*(p^{-m}\epsilon)$ is representable by a perfect flat formal scheme over $\mathbb{F}_p[[t^{1/(p-1)p^\infty}]]$ which is naturally the same as $\mathfrak{X}'^*(\epsilon)^{\mathrm{perf}}$.
- Its generic fiber is thus a perfectoid space over $\mathbb{F}_p((t^{1/(p-1)p^\infty}))$, that is identified with the open subset of $\mathcal{X}'^{*\mathrm{perf}}$ where $|\mathrm{Ha}| \geq |t|^\epsilon$.
- We have a canonical identification
$$\mathfrak{X}'^*(p^{-m}\epsilon)/t^{1-\epsilon} \simeq \mathfrak{X}^*(p^{-m}\epsilon)/p^{1-\epsilon}$$
compatible with transition maps.
- For an open affine $\mathrm{Spf}(R_{m_0}) \subset \mathfrak{X}^*(p^{-m}\epsilon)$ with preimages $\mathrm{Spf}(R_m)$, we get affine opens $\mathrm{Spf}(S_m) \subset \mathfrak{X}'^*(p^{-m}\epsilon)$, with
$$S_m/t^{1-\epsilon} = R_m/p^{1-\epsilon}.$$
- Let R_∞ be the p-adic completion of $\mathrm{colim}_m R_m$. Let S_∞ be the t-adic completion of $\mathrm{colim}_m S_m$.
- Then $\mathrm{Spf}(R_\infty) \subset \mathfrak{X}_{\Gamma_s(p^\infty)}(\epsilon)_a$ and $\mathrm{Spf}(S_\infty) \subset \mathfrak{X}'^*(\epsilon)^{\mathrm{perf}}$ give corresponding open subsets, and
$$R_\infty/p^{1-\epsilon} = \mathrm{colim}_m R_m/p^{1-\epsilon} = \mathrm{colim}_m S_m/t^{1-\epsilon} = S_\infty/t^{1-\epsilon}.$$
- It follows that $R_\infty[1/p]$ and $S_\infty[1/t]$ are tilts by [Sch12, Theorem 5.2].

□

Lemma 4.2.10. The space $\mathcal{X}^*_{\Gamma_s(p^\infty)}(\epsilon)_a$ is affinoid.

Proof. ((todo: use tilting))

- It suffices to check for the tilts.
- The open subset $\mathcal{X}'^*(\epsilon) \subset \mathcal{X}'^*$ given by $|\mathrm{Ha}| \geq |\epsilon|^\epsilon$ is affinoid.

□

4.3. **Lifting to level Γ_1.**

4.3.1. *Specialized version of Tate's normalized trace.*

Lemma 4.3.1. Let $\mathfrak{X}_{\Gamma_s(p^\infty)}(\epsilon)_a$ be the formal scheme over $\mathrm{Spf}(\mathbb{Z}_p^{\mathrm{cycl}})$ defined as

$$\mathfrak{X}_{\Gamma_s(p^\infty)}(\epsilon)_a = \lim_m \mathfrak{X}(p^{-m}\epsilon).$$

Let $0 \leq m \leq m'$. Then

(1) The maps
$$1/p^{(m'-m)g(g+1)/2}\mathrm{tr} : \mathcal{O}_{\mathfrak{X}(p^{-m'}\epsilon)}[1/p] \to \mathcal{O}_{\mathfrak{X}(p^{-m}\epsilon)}[1/p]$$
are compatible for varying m', and thus induces a map
$$\overline{\mathrm{tr}}_m : \lim_{m'} \mathcal{O}_{\mathfrak{X}(p^{-m'}\epsilon)}[1/p] \to \mathcal{O}_{\mathfrak{X}(p^{-m}\epsilon)}[1/p].$$
(2) The image of $\overline{\mathrm{tr}}_m$ is contained in $p^{-C_m}\mathcal{O}_{\mathfrak{X}(p^{-m}\epsilon)}$ for some constant C_m, with $C_m \to 0$ as $m \to +\infty$. Thus $\overline{\mathrm{tr}}_m$ extends by continuity to a map
$$\overline{\mathrm{tr}}_m : \mathcal{O}_{\mathfrak{X}_{\Gamma_s(p^\infty)}(\epsilon)_a}[1/p] \to \mathcal{O}_{\mathfrak{X}(p^{-m}\epsilon)}[1/p],$$
called Tate's normalized trace.
(3) For every $x \in \mathcal{O}_{\mathfrak{X}_{\Gamma_s(p^\infty)}(\epsilon)_a}[1/p]$, we have
$$x = \lim_{m\to+\infty} \overline{\mathrm{tr}}_m(x).$$

Proof. Omitted. □

4.3.2. *A general result.*

Situation 4.3.2. Let an integer $m \geq 1$ which is sufficiently large such that $\mathcal{X}^*_{\Gamma_s(p^m)}(\epsilon)_a$ is affinoid, cf. Lemma 4.2.7. Let $\mathcal{Y}^*_m \to \mathcal{X}^*_{\Gamma_s(p^m)}(\epsilon)_a$ be a finite morphism. Let $\mathcal{Y}_m \to \mathcal{X}_{\Gamma_s(p^m)}(\epsilon)_a$ be the pullback of $\mathcal{Y}^*_m \to \mathcal{X}^*_{\Gamma_s(p^m)}(\epsilon)_a$ along $\mathcal{X}_{\Gamma_s(p^m)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_s(p^m)}(\epsilon)_a$. Assume that

(1) The map $\mathcal{Y}_m \to \mathcal{X}_{\Gamma_s(p^m)}(\epsilon)_a$ is finite étale.
(2) $\mathcal{Y}^*_m$ is normal.
(3) None of the irreducible components of $\mathcal{Y}^*_m$ is mapped into the boundary of $\mathcal{X}^*_{\Gamma_s(p^m)}(\epsilon)_a$.

For $m' \geq m$, define $\mathcal{Y}^*_{m'} \to \mathcal{X}^*_{\Gamma_s(p^{m'})}(\epsilon)_a$ to be the ((todo: normalization??)) pullback of $\mathcal{Y}^*_m \to \mathcal{X}^*_{\Gamma_s(p^m)}(\epsilon)_a$ along $\mathcal{X}^*_{\Gamma_s(p^{m'})}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_s(p^m)}(\epsilon)_a$. Define $\mathcal{Y}_{m'} \to \mathcal{X}_{\Gamma_s(p^{m'})}(\epsilon)_a$ by pullback. Let $\mathcal{Y}_\infty$ be the pullback of $\mathcal{Y}_m$ to $\mathcal{X}_{\Gamma_s(p^\infty)}(\epsilon)_a$, which exists as $\mathcal{Y}_m \to \mathcal{X}_{\Gamma_s(p^m)}(\epsilon)_a$ is finite étale.

Since every $\mathcal{X}^*_{\Gamma_s(p^{m'})}(\epsilon)_a$ is affinoid, each $\mathcal{Y}^*_{m'}$ is affinoid. We write $\mathcal{Y}^*_{m'} = \mathrm{Spa}(S_{m'}, S^+_{m'})$.

((todo: scholze says $S^+_{m'} = S^\circ_{m'}$??))

Lemma 4.3.3. In the Situation 4.3.2, we have

(1) For all $m' \geq m$,
$$S^+_{m'} = H^0(\mathcal{Y}_{m'}, \mathcal{O}^+_{\mathcal{Y}_{m'}}).$$
(2) The map
$$\mathrm{colim}_{m'} S^+_{m'} \to H^0(\mathcal{Y}_\infty, \mathcal{O}^+_{\mathcal{Y}_\infty})$$
is injective with dense image. Moreover, there is a canonical continuous retraction
$$H^0(\mathcal{Y}_\infty, \mathcal{O}_{\mathcal{Y}_\infty}) \to S_{m'}.$$

(3) Assume that $S_\infty = H^0(\mathcal{Y}_\infty, \mathcal{O}_{\mathcal{Y}_\infty})$ is a perfectoid $\mathbb{Q}_p^{\mathrm{cycl}}$-algebra. Then
$$\mathcal{Y}_\infty^* = \mathrm{Spa}(S_\infty, S_\infty^+)$$
where $S_\infty^+ = S_\infty^\circ$, is an affinoid perfectoid space over $\mathbb{Q}_p^{\mathrm{cycl}}$, and
$$\mathcal{Y}_\infty^* \sim \lim_{m'} \mathcal{Y}_{m'}^*,$$
and S_∞^+ is the p-adic completion of $\mathrm{colim}_{m'} S_{m'}^+$.

Proof. Proof of (1). By replacing m with m', it sufficies to prove the claim for $m' = m$. The desired isomorphism is automatic if we have the following isomorphism
$$S_m \simeq H^0(\mathcal{Y}_m, \mathcal{O}_{\mathcal{Y}_m}).$$
Write $R = H^0(\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a, \mathcal{O}_{\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a})$. By the assumption, the map $R \to S_m$ is finite and étale away from boundary (recall that m is sufficiently large such that $\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a$ is affinoid). Let $Z \subset \mathrm{Spec}(R)$ be the boundary, which is of codimension ≥ 2. Then the preimage $Z' \subset \mathrm{Spec}(S_m)$ is also of codimension ≥ 2 by Condition (3) in Situation 4.3.2. Both S_m and R are normal and Noetherian. Hence Lemma 2.2.4 shows that
$$S_m = H^0(\mathrm{Spec}(S_m)\backslash Z', \mathcal{O}_{\mathrm{Spec}(S_m)}), \quad R = H^0(\mathrm{Spec}(R)\backslash Z, \mathcal{O}_{\mathrm{Spec}(R)}).$$
Since the map $R \to S_m$ is finite étale away from boundary, we have a trace map $\mathcal{O}_{\mathrm{Spec}(S_m)}|_{\mathrm{Spec}(S_m)\backslash Z'} \to \mathcal{O}_{\mathrm{Spec}(R)}|_{\mathrm{Spec}(R)\backslash Z}$. Taking global sections and identifying using the two isomorphisms above, we obtain the map $\mathrm{tr}_{S_m/R} : S_m \to R$. The next claim is that the associated pairing
$$S_m \otimes_R S_m \to R, \quad s_1 \otimes s_2 \mapsto \mathrm{tr}_{S_m/R}(s_1 s_2)$$
induces an isomorphism $S_m \simeq \mathrm{Hom}_R(S_m, R)$. To see this, let $s_1 \in S_m$ be an element lying in the kernel. Then it lies in the kernel of the pairing away from the boundary, on which it is perfect as $R \to S_m$ is finite étale away from the boundary. Hence $s_1 = 0$ away from the boundary, and thus is zero (by Hartog's extension principle, again). Similarly, any element of $\mathrm{Hom}_R(S_m, R)$ comes from a unique eleement of S_m away from the boundary, and thus from an element of S_m.

For an affinoid open subset $\mathcal{U}$ of $\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a$ with preimage $\mathcal{V} \subset \mathcal{Y}_m^*$, repeat the argument above, and we obtain an isomorphism
$$H^0(\mathcal{V}, \mathcal{O}_{\mathcal{Y}_m^*}) \simeq \mathrm{Hom}_R(S_m, H^0(\mathcal{U}, \mathcal{O}_{\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a})).$$
These isomorphisms can be glued such that the same isomorphism holds for every open subset $\mathcal{U} \subset \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a$. Take $\mathcal{U} = \mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a$ and we are done.

Proof of (2). We need to show that the map $\mathrm{colim}_{m'} H^0(\mathcal{Y}_{m'}, \mathcal{O}^+_{\mathcal{Y}_{m'}}) \to H^0(\mathcal{Y}_\infty, \mathcal{O}^+_{\mathcal{Y}_\infty})$ is injective. ((todo))

Proof of (3). This is a direct corollary of (2). ☐

4.3.3. *xxx.*

Definition 4.3.4. Note that on the tower $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a$, we have the tautological Abelian variety $\mathcal{A}^t_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a$ (which are related to each other by pullback), as well as the Abelian varieties $\mathcal{A}_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a = \mathcal{A}(p^{-m}\epsilon)$ over $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a \simeq \mathcal{X}(p^{-m}\epsilon)$. They are related by an isogeny
$$\mathcal{A}_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a \to \mathcal{A}^t_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a$$
whose kernel is the canonical subgroup $C_m \subset \mathcal{A}_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a[p^m]$ of level m. We get an induced subgroup
$$D_m = \mathcal{A}_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a[p^m]/C_m \subset \mathcal{A}^t_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a.$$
Let $D_{m\Gamma_{\mathrm{s}}(p^{m'})}$ be the pullback of D_m to $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^{m'})}(\epsilon)_a$ for $m' \geq m$. We have
$$D_{m\Gamma_{\mathrm{s}}(p^{m'})} = D_{m'}[p^{m'}].$$
Also, the D_m give the $\Gamma_{\mathrm{s}}(p^m)$ level structure. Let $D_{m\Gamma_{\mathrm{s}}(p^\infty)}$ be the pullback of D_m to $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\epsilon)_a$. Since $D_m \to \mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a$ is finite étale, $D_{m\Gamma_{\mathrm{s}}(p^\infty)}$ is a perfectoid space.

Lemma 4.3.5. The map
$$\mathcal{A}_{\Gamma_{\mathrm{s}}(p^\infty)}(\epsilon)_a[p^m] \to D_{m\Gamma_{\mathrm{s}}(p^\infty)}$$
is an isomorphism of perfectoid spaces.

Proof. Let (R, R^+) be a perfectoid affinoid $\mathbb{Q}_p^{\mathrm{cycl}}$-algebra. Then

$$\mathcal{A}_{\Gamma_{\mathrm{s}}(p^\infty)}(\epsilon)_a[p^m](R, R^+) = \lim_{m'} \mathcal{A}_{\Gamma_{\mathrm{s}}(p^{m'})}(\epsilon)_a[p^m](R, R^+)$$

The transition map

$$\mathcal{A}_{\Gamma_{\mathrm{s}}(p^{m'+m})}(\epsilon)_a[p^m] \to \mathcal{A}_{\Gamma_{\mathrm{s}}(p^{m'})}(\epsilon)_a[p^m]$$

kills the canonical subgroup C_m of level m (of $\mathcal{A}_{\Gamma_{\mathrm{s}}(p^{m'+m})}(\epsilon)_a$), so it factors as

$$\mathcal{A}_{\Gamma_{\mathrm{s}}(p^{m'+m})}(\epsilon)_a[p^m] \to \mathcal{A}_{\Gamma_{\mathrm{s}}(p^{m'+m})}(\epsilon)_a[p^m]/C_m = D_{m\Gamma_{\mathrm{s}}(p^{m'+m})} \to \mathcal{A}_{\Gamma_{\mathrm{s}}(p^{m'})}(\epsilon)_a[p^m].$$

This shows that the desired isomorphism holds, since

$$D_{m\Gamma_{\mathrm{s}}(p^\infty)}(R, R^+) = \lim_{m'} D_{m\Gamma_{\mathrm{s}}(p^{m'})}(R, R^+).$$

□

Definition 4.3.6. Let D'_m be the quotient $\mathcal{A}'(\epsilon)[p^m]/C'_m$, where C'_m denotes the canonical subgroup of level m of $\mathcal{A}'(\epsilon)$. We have $D'_m \to \mathcal{X}'(\epsilon) \subset \mathcal{X}'$. Note that all Abelian varieties over $\mathbb{F}_p((t^{1/(p-1)p^\infty}))$ parametrized by $\mathcal{X}'(\epsilon)$ are ordinary, as the Hasse invariant divides t^ϵ which is invertible.

Lemma 4.3.7. The tilt of $D_{m\Gamma_{\mathrm{s}}(p^\infty)}$ identifies canonically with the perfection of D'_m.

Proof. Recall from the uniqueness of canonical subgroups (cf. Lemma 2.1.5) that

$$C'_m(R') = \{s \in \mathcal{A}'(\epsilon)[p^m](R') \mid s \equiv 0 \bmod p^{(1-\epsilon)/p^m}\}.$$

So C'_m is the killed by the Frobenius map. Thus passing to perfection kills C'_m, and hence

$$D'^{\mathrm{perf}}_m = \mathcal{A}'(\epsilon)[p^m]^{\mathrm{perf}}.$$

Recall that $\mathcal{A}_{\Gamma_{\mathrm{s}}(p^\infty)}(\epsilon)_a^\flat \simeq \mathcal{A}'(\epsilon)^{\mathrm{perf}}$, cf. Lemma 4.2.9, we conclude that

$$D'^{\mathrm{perf}}_m = \mathcal{A}_{\Gamma_{\mathrm{s}}(p^\infty)}(\epsilon)_a[p^m]^\flat.$$

Finally, combine with Lemma 4.3.5 and we obtain

$$D'^{\mathrm{perf}}_m = D^\flat_{m\Gamma_{\mathrm{s}}(p^\infty)}.$$

□

Definition 4.3.8. Let $\mathcal{X}'^*_{\Gamma_1(p^m)}(\epsilon)$ be the open locus of the adic space associated with

$$X^{\mathrm{ord}*}_{\Gamma_1(p^m)} \otimes_{\mathbb{F}_p} \mathbb{F}_p((t^{1/(p-1)p^\infty}))$$

where $|\mathrm{Ha}| \geq |t|^\epsilon$. Then

$$\mathcal{X}'^*_{\Gamma_1(p^m)}(\epsilon) \to \mathcal{X}'^*(\epsilon)$$

is finite, and étale away from the boundary. In particular, the base-change $\mathcal{X}'_{\Gamma_1(p^m)}(\epsilon) \to \mathcal{X}'(\epsilon) \subset \mathcal{X}'^*(\epsilon)$ is finite étale, parametrizing isomorphisms $D'_m \simeq (\mathbb{Z}/p^m\mathbb{Z})^g$. Let $\mathcal{Z}'^*(\epsilon) \subset \mathcal{X}'^*(\epsilon)$ denote the boundary, with pullback $\mathcal{Z}'^*_{\Gamma_1(p^m)}(\epsilon) \subset \mathcal{X}'^*_{\Gamma_1(p^m)}(\epsilon)$.

Lemma 4.3.9. The triple $(\mathcal{X}'^*(\epsilon)^{\mathrm{perf}}, \mathcal{Z}'^*(\epsilon)^{\mathrm{perf}}, \mathcal{X}'(\epsilon)^{\mathrm{perf}})$ is good, cf. Definition 2.4.1.

Proof. Recall that $\mathcal{X}'^*(\epsilon)$ is the generic fiber of the formal scheme $\mathfrak{X}'^*$. In the light of Lemma 2.4.5, it sufficies to prove the similar result after restricting to every affine open of $\mathfrak{X}'^*$, and this is given by Lemma 2.4.6. Note that $X^* \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p$ admits a resolution of singularities given by the toroidal compactification, cf. [FC13]. □

Lemma 4.3.10. The triple $(\mathcal{X}'^*_{\Gamma_1(p^m)}(\epsilon)^{\mathrm{perf}}, \mathcal{Z}'^*_{\Gamma_1(p^m)}(\epsilon)^{\mathrm{perf}}, \mathcal{X}'_{\Gamma_1(p^m)}(\epsilon)^{\mathrm{perf}})$ is good.

Proof. Combine Lemma 4.3.9 and Lemma 2.4.4. □

4.3.4. *Back to the tower.* Now fix $m \geq 1$, and consider $\mathcal{Y}_m^* = \mathcal{X}^*_{\Gamma_1(p^m)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_0(p^m)}(\epsilon)_a$.

In the following, we denote $\mathcal{Y}_m^* = \mathcal{X}^*_{\Gamma_1(p^m)}(\epsilon)_a$.

Lemma 4.3.11. The tilt of $\mathcal{Y}_\infty$ identifies with $\mathcal{X}'_{\Gamma_1(p^m)}(\epsilon)^{\mathrm{perf}}$.

Proof. Recall that the map $\mathcal{X}^*_{\Gamma_1(p^m)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_0(p^m)}(\epsilon)_a$ is finite étale, and thus the base-change $\mathcal{Y}_m \to \mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a$ is also finite étale. Hence the map $\mathcal{Y}_\infty \to \mathcal{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\epsilon)_a$ is finite étale as it is a pullback of $\mathcal{Y}_m$. Recall from the moduli interpretations that the map $\mathcal{Y}_\infty \to \mathcal{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\epsilon)_a$ parametrizes all isomorphisms $D_{m\Gamma_{\mathrm{s}}(p^\infty)} \simeq (\mathbb{Z}/p^m\mathbb{Z})^g$ ((todo: ref)). Apply Lemma 4.3.7, we see that the tilt of $\mathcal{Y}_\infty$ parametrizes all isomorphism $D'^{\mathrm{perf}}_m \simeq (\mathbb{Z}/p^m\mathbb{Z})^g$. Therefore the tilt of $\mathcal{Y}_m$ identifies with $\mathcal{X}'_{\Gamma_1(p^m)}(\epsilon)^{\mathrm{perf}}$, cf. Lemma 4.2.9. □

Remark 4.3.12. Note that $\mathcal{Y}^*_m\backslash\partial \to \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\epsilon)_a\backslash\partial$ is finite étale. By pullback we get a perfectoid space $\mathcal{Y}^*_\infty\backslash\partial \to \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^\infty)}(\epsilon)_a\backslash\partial$. We warn the reader that $\mathcal{Y}^*_\infty$ is not defined yet.

Lemma 4.3.13. The tilt of $\mathcal{Y}^*_\infty\backslash\partial$ identifies with $\mathcal{X}'^*_{\Gamma_1(p^m)}(\epsilon)^{\mathrm{perf}}\backslash\partial$.

Proof. Let $\mathcal{X}'^*_{\Gamma_1(p^m)}(\epsilon)^{\mathrm{perf}} = \mathrm{Spa}(T, T^+)$, cf. Lemma 4.3.10. Let (U, U^+) be the untilt of (T, T^+). Using Lemma 4.3.11 and taking global sections, we obtain a map

$$U^+ \to H^0(\mathcal{Y}_\infty, \mathcal{O}^+_{\mathcal{Y}_\infty}) = S^+_\infty.$$

Recall that S^+_∞ is the p-adic completion of $\mathrm{colim}_{m'} S^+_{m'}$, cf. Lemma 4.3.3. Thus we have a map of adic spaces $\mathcal{Y}^*_m\backslash\partial \to \mathrm{Spa}(S_\infty, S^+_\infty)$. Combining the two maps and $\mathcal{Y}^*_\infty\backslash\partial \to \mathcal{Y}^*_m\backslash\partial$, we get $\mathcal{Y}^*_\infty\backslash\partial \to \mathrm{Spa}(U, U^+)$. The two spaces are finite étale over $\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^\infty)}(\epsilon)_a$ away from boundary, cf. Lemma 4.2.9, Definition 4.3.8, and Remark 4.3.12. Finally, we can apply Lemma B.0.3 with Lemma 4.3.9 and Lemma 4.3.11, i.e. to the following diagram

$$\begin{array}{ccc}
\mathcal{Y}^\flat_\infty & \xrightarrow{\sim} & \mathcal{X}'_{\Gamma_1(p^m)}(\epsilon)^{\mathrm{perf}} \\
\downarrow & & \downarrow \\
(\mathcal{Y}^*_\infty\backslash\partial)^\flat & \longrightarrow & \mathcal{X}'^*_{\Gamma_1(p^m)}(\epsilon)^{\mathrm{perf}}\backslash\partial \\
& \searrow \quad \swarrow & \\
& (\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^\infty)}(\epsilon)_a\backslash\partial)^\flat \simeq \mathcal{X}'^*(\epsilon)^{\mathrm{perf}}\backslash\partial. &
\end{array}$$

Therefore the tilt of $\mathcal{Y}^*_\infty\backslash\partial$ identifies with $\mathcal{X}'^*_{\Gamma_1(p^m)}(\epsilon)^{\mathrm{perf}}\backslash\partial$. □

Lemma 4.3.14. The ring $S_\infty = H^0(\mathcal{Y}_\infty, \mathcal{O}_{\mathcal{Y}_\infty})$ is perfectoid, and the tilt of $\mathcal{Y}^*_\infty = \mathrm{Spa}(S_\infty, S^+_\infty)$ identifies with $\mathcal{X}'^*_{\Gamma_1(p^m)}(\epsilon)^{\mathrm{perf}}$.

Proof. Let $\mathcal{X}'^*_{\Gamma_1(p^m)}(\epsilon)^{\mathrm{perf}} = \mathrm{Spa}(T, T^+)$, cf. Lemma 4.3.10. Let (U, U^+) be the untilt of (T, T^+). Using Lemma 4.3.11 and taking global sections, we obtain a map

$$U^+ \to H^0(\mathcal{Y}_\infty, \mathcal{O}^+_{\mathcal{Y}_\infty}) = S^+_\infty.$$

It sufficies to show that this map is an isomorphism. It's clear that we have an injection $S^+_\infty/p \to H^0(\mathcal{Y}_\infty, \mathcal{O}^+_{\mathcal{Y}_\infty}/p)$. Hence the map

$$(U^+/p)^a = H^0(\mathcal{X}'^*_{\Gamma_1(p^m)}(\epsilon)^{\mathrm{perf}}, \mathcal{O}^+/t)^a \to H^0(\mathcal{X}'_{\Gamma_1(p^m)}(\epsilon)^{\mathrm{perf}}, \mathcal{O}^+/t)^a) = H^0(\mathcal{Y}_\infty, \mathcal{O}^+/p)^a = (S^+_\infty/p)^a$$

is injective, cf. Lemma 4.3.10. So $U^+ \to S^+_\infty$ is injective.

Now we prove the surjectivity. We have a map

$$(S^+_\infty/p)^a \to H^0(\mathcal{Y}^*_\infty\backslash\partial, \mathcal{O}^+/p)^a \simeq H^0(\mathcal{X}'^*_{\Gamma_1(p^m)}(\epsilon)^{\mathrm{perf}}\backslash\partial, \mathcal{O}^+/t)^a \simeq H^0(\mathcal{X}'^*_{\Gamma_1(p^m)}(\epsilon)^{\mathrm{perf}}, \mathcal{O}^+/t)^a = (U^+/p)^a,$$

where the first isomorphism is provided by Lemma 4.3.13, and the second isomorphism is provided by Lemma 4.3.10. Hence the proof is complete. □

Lemma 4.3.15. For any $m \geq 1$, there is a unique perfectoid spacae $\mathcal{X}^*_{\Gamma_1(p^m)\cap\Gamma_s(p^\infty)}(\epsilon)_a$ over $\mathbb{Q}_p^{\mathrm{cycl}}$ such that

$$\mathcal{X}^*_{\Gamma_1(p^m)\cap\Gamma_s(p^\infty)}(\epsilon)_a \sim \lim_{m'} \mathcal{X}^*_{\Gamma_1(p^m)\cap\Gamma_s(p^{m'})}(\epsilon)_a.$$

Moreover, $\mathcal{X}^*_{\Gamma_1(p^m)\cap\Gamma_s(p^\infty)}(\epsilon)_a$ and all $\mathcal{X}^*_{\Gamma_1(p^m)\cap\Gamma_s(p^{m'})}(\epsilon)_a$ for m' sufficiently large are affinoid, and

$$\mathrm{colim}_{m'} H^0(\mathcal{X}^*_{\Gamma_1(p^m)\cap\Gamma_s(p^{m'})}(\epsilon)_a, \mathcal{O}) \to H^0(\mathcal{X}^*_{\Gamma_1(p^m)\cap\Gamma_s(p^\infty)}(\epsilon)_a, \mathcal{O})$$

has dense image. Let $\mathcal{Z}^*_{\Gamma_1(p^m)\cap\Gamma_s(p^\infty)}(\epsilon)_a \subset \mathcal{X}^*_{\Gamma_1(p^m)\cap\Gamma_s(p^\infty)}(\epsilon)_a$ denote the boundar, and $\mathcal{X}_{\Gamma_1(p^m)\cap\Gamma_s(p^\infty)}(\epsilon)_a$ the preimage of $\mathcal{X}_{\Gamma_s(p)}(\epsilon)_a \subset \mathcal{X}_{\Gamma_s(p)}(\epsilon)_a$. Then the triple

$$(\mathcal{X}^*_{\Gamma_1(p^m)\cap\Gamma_s(p^\infty)}(\epsilon)_a, \mathcal{Z}_{\Gamma_1(p^m)\cap\Gamma_s(p^\infty)}(\epsilon)_a, \mathcal{X}_{\Gamma_1(p^m)\cap\Gamma_s(p^\infty)}(\epsilon)_a)$$

is good.

Proof. This follows directly from Lemma 4.3.3 and Lemma 4.3.14. □

Lemma 4.3.16. There is a unique perfectoid space $\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$ over $\mathbb{Q}_p^{\mathrm{cycl}}$ such that

$$\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a \sim \lim_m \mathcal{X}^*_{\Gamma_1(p^m)}(\epsilon)_a.$$

Moreover, $\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$ and all $\mathcal{X}^*_{\Gamma_1(p^m)}(\epsilon)_a$ for m sufficiently large are affinoid, and

$$\mathrm{colim}_m H^0(\mathcal{X}^*_{\Gamma_1(p^m)}(\epsilon)_a, \mathcal{O}) \to H^0(\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a, \mathcal{O})$$

has dense image. Let $\mathcal{Z}^*_{\Gamma_1(p^\infty)}(\epsilon)_a \subset \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$ denote the boundar, and $\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a$ the preimage of $\mathcal{X}_{\Gamma_s(p)}(\epsilon)_a \subset \mathcal{X}_{\Gamma_s(p)}(\epsilon)_a$. Then the triple

$$(\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a, \mathcal{Z}_{\Gamma_1(p^\infty)}(\epsilon)_a, \mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a)$$

is good.

Proof. Pass to the limit on m in Lemma 4.3.15 for the first claim. Apply Lemma 4.3.3 for the second result. Finally use Lemma 2.4.5 for the last assertion. □

4.4. **Lifting to level Γ.**

Lemma 4.4.1. For every $m \geq 1$, the map

$$\mathcal{X}^*_{\Gamma(p^m)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_1(p^m)}(\epsilon)_a$$

is finite étale.

Proof. First take $\epsilon = 0$. We claim that we have a decomposition

$$\mathcal{X}^*_{\Gamma(p^m)}(0)_a \simeq \bigsqcup_{\Gamma_1(p^m)/\Gamma(p^m)} \mathcal{X}^*_{\Gamma_1(p^m)}(0)_a.$$

((todo)) □

Lemma 4.4.2. There is a unique perfectoid space $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ over $\mathbb{Q}_p^{\mathrm{cycl}}$ such that

$$\mathcal{X}^*_{\Gamma(p^m)}(\epsilon)_a \sim \lim_m \mathcal{X}^*_{\Gamma(p^m)}(\epsilon)_a.$$

Moreover, $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ and all $\mathcal{X}^*_{\Gamma(p^m)}(\epsilon)_a$ for m sufficiently large are affinoid. The triple

$$(\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a, \mathcal{Z}_{\Gamma(p^\infty)}(\epsilon)_a, \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a)$$

is good.

Proof. This follows from Lemma 4.4.1, Lemma 4.3.16, and almost purity. □

5. The Hodge–Tate period map

5.1. The map of topological spaces.

Definition 5.1.1. The topological space $|\mathcal{X}^*_{\Gamma(p^\infty)}|$ (resp. $|\mathcal{X}_{\Gamma(p^\infty)}|$, $|\mathcal{Z}_{\Gamma(p^\infty)}|$) is defined as the limit $\lim_m|\mathcal{X}^*_{\Gamma(p^m)}|$ (resp. $\lim_m|\mathcal{X}_{\Gamma(p^m)}|$, $\lim_m|\mathcal{Z}_{\Gamma(p^m)}|$).

Definition 5.1.2. Let K be a complete non-Archimedean field extension of $\mathbb{Q}_p^{\mathrm{cycl}}$. Let $K^+ \subset K$ be an open and bounded valuation subring. We define

$$\mathcal{X}^*_{\Gamma(p^\infty)}(K, K^+) = \lim_m \mathcal{X}^*_{\Gamma(p^m)}(K, K^+).$$

Construction 5.1.3. We now construct the map

$$|\pi_{\mathrm{HT}}| : |\mathcal{X}^*_{\Gamma(p^\infty)}|\backslash|\mathcal{Z}^*_{\Gamma(p^\infty)}| \to |\mathscr{F}\!\ell|.$$

Let $x \in |\mathcal{X}^*_{\Gamma(p^\infty)}|\backslash|\mathcal{Z}^*_{\Gamma(p^\infty)}|$ be a point. Denote its image by $x_m \in |\mathcal{X}^*_{\Gamma(p^m)}|\backslash|\mathcal{Z}^*_{\Gamma(p^m)}|$ for $m \geq 1$. The point x_m induces a map $\phi_m : \mathrm{Spa}(L_m, L_m^+) \to \mathcal{X}^*_{\Gamma(p^m)}\backslash\mathcal{Z}_{\Gamma(p^m)} = X^{\mathrm{ad}}_{\Gamma(p^m),\mathbb{Q}_p^{\mathrm{cycl}}}$, cf. Lemma B.0.4. Note that L_m is the completion of the residue field of x_m, and (L_m, L_m^+) is a non-discrete affinoid field. The extension L_{m+1}/L_m is finite, and thus there exists a unique minimal non-discrete affinoid field (L, L^+) containing all of (L_m, L_m^+). Hence we get a map $\phi_m : \mathrm{Spa}(L, L^+) \to X^{\mathrm{ad}}_{\Gamma(p^m),\mathbb{Q}_p^{\mathrm{cycl}}}$ that factors through $\mathrm{Spa}(L_m, L_m^+)$. Restricting to the unique generic point $\mathrm{Spa}(L, \mathcal{O}_L)$, we get a map of schemes $\mathrm{Spec}(L) \to X_{\Gamma(p^m),\mathbb{Q}_p^{\mathrm{cycl}}}$. The moduli interpretation gives a principally polarized Abelian variety A over L of dimension g, with level structure $\eta_p : A[p^m](\overline{L}) \to (\mathbb{Z}/p^m\mathbb{Z})^{2g}$. We then get a map $\eta_p : T_p(A_{\overline{L}}) \to \mathbb{Z}_p^{2g}$ by varying m. Let $C = \widehat{\overline{L}}$ be the completion. Then we have the Hodge–Tate filtration

$$0 \to \mathrm{Lie}(A_C) \to T_p(A_C) \otimes_{\mathbb{Z}_p} C \simeq C^{2g}.$$

In other words, we obtain a g-dimensional subspace of C^{2g}, which then gives an L-point of Fl, and finally we obtains an $\mathrm{Spa}(L, \mathcal{O}_L)$-point of $\mathscr{F}\!\ell$. Since Fl is proper, it extends to a map $\mathrm{Spa}(L, L^+) \to \mathscr{F}\!\ell$.

Lemma 5.1.4. There is a $G(\mathbb{Q}_p)$-equivariant continuous map

$$|\pi_{\mathrm{HT}}| : |\mathcal{X}^*_{\Gamma(p^\infty)}|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}| \to |\mathscr{F}\!\ell|,$$

defined by sending a point $x \in (\mathcal{X}^*_{\Gamma(p^\infty)}\backslash\mathcal{Z}_{\Gamma(p^\infty)})(K, K^+)$ correpsonding to a principally polarized Ablian variety A/K and a symplectic isomorphism $\alpha : T_pA \to \mathbb{Z}_p^{2g}$, to the Hodge–Tate filtration $\mathrm{Lie}(A) \subset K^{2g}$.

Proof. We show the continuity. Let $S = \mathcal{X}^*\backslash\mathcal{Z} = X^{\mathrm{ad}}_{\mathbb{Q}_p^{\mathrm{cycl}}}$ be an adic space. Let $g : A_S \to S$ be the universal Abelian variety. By [Sch13a, Theorem 1.3], we have almost ismorphisms

$$(R^1g_*\mathbb{Z}/p^n\mathbb{Z}) \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \mathcal{O}_S^+/p^n \to R^1g_*\mathcal{O}^+_{A_S}/p^n$$

for all $n \geq 1$. Passing to the limit over $n \geq 1$, we obtain an isomorphism

$$R^1g_*\widehat{\mathbb{Z}}_p \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathcal{O}}_S \to R^1g_*\widehat{\mathcal{O}}_{A_S}$$

of sheaves on the pro-étale site $S_{\text{proét}}$. Hence we get a map

$$(R^1g_*\mathcal{O}_{A_S}) \otimes_{\mathcal{O}_S} \widehat{\mathcal{O}}_S \to R^1g_*\widehat{\mathcal{O}}_{A_S}.$$

The adic space S can be covered by affinoid open subsets $\{U_i\}$, such that each U_i is the bottom level of an affinoid perfectoid object $(U_{i,j})_{j\in J}$ in $S_{\text{proét}}$ such that all transition maps are finite étale surjective. Let $\widehat{U}_{i,m}$ be the base change of $\widehat{U}_i$ from S to $\mathcal{X}^*_{\Gamma(p^m)}\backslash\mathcal{Z}_{\Gamma(p^m)}$. By almost purity, $\widehat{U}_{i,m}$ is affinoid perfectoid, as $\widehat{U}_i$ is affinoid perfectoid. Hence we have an affinoid perfectoid space

$$\widehat{U}_{i,\infty} \sim \lim_m \widehat{U}_{i,m},$$

which is also the affinoid perfectoid space associated to the following affinoid perfectoid object in $S_{\text{proét}}$

$$V = (U_{i,j} \times_S \mathcal{X}^*_{\Gamma(p^m)}\backslash\mathcal{Z}_{\Gamma(p^m)})_{j\in J, m\geq 1}.$$

Evaluating the map

$$(R^1g_*\mathcal{O}_{A_S}) \otimes_{\mathcal{O}_S} \widehat{\mathcal{O}}_S \to R^1g_*\widehat{\mathbb{Z}}_p \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathcal{O}}_S$$

at V gives a map

$$\mathrm{Lie}(A_S)\otimes_{\mathcal{O}_S}\mathcal{O}_V\to\mathcal{O}_V^{2g}$$

which is injective, and the image is totally isotropic. Hence we obtain a map of adic spaces

$$\widehat{U}_{i,\infty}\to\mathscr{F}\!\ell.$$

We have the following commutative diagram

$$\begin{array}{ccc} |\widehat{U}_{i,\infty}| & \longrightarrow & |\mathscr{F}\!\ell| \\ \downarrow & & \uparrow \\ |U_i|\times_{|S|}(|\mathcal{X}^*_{\Gamma(p^\infty)}|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}|) & \longrightarrow & |\mathcal{X}^*_{\Gamma(p^\infty)}|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}|. \end{array}$$

The map $|\widehat{U}_{i,\infty}|\to|U_i|\times_{|S|}(|\mathcal{X}^*_{\Gamma(p^\infty)}|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}|)$ on the left is continuous, open, and surjective. Hence the restriction $|U_i|\times_{|S|}(|\mathcal{X}^*_{\Gamma(p^\infty)}|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}|)\subset|\mathcal{X}^*_{\Gamma(p^\infty)}|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}|\to|\mathscr{F}\!\ell|$ is continuous. Therefore the map $|\pi_{\mathrm{HT}}|:|\mathcal{X}^*_{\Gamma(p^\infty)}|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}|\to|\mathscr{F}\!\ell|$ is continuous. □

Lemma 5.1.5. The preimage of $\mathscr{F}\!\ell(\mathbb{Q}_p)\subset|\mathscr{F}\!\ell|$ under $|\pi_{\mathrm{HT}}|$ is given by the closure of

$$|\mathcal{X}^*_{\Gamma(p^\infty)}(0)|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}(0)|.$$

Proof. First, note that $|\mathcal{X}^*_{\Gamma(p^\infty)}(0)|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}(0)|$ is a retro-compact open subset of the locally spectral topological space $|\mathcal{X}^*_{\Gamma(p^\infty)}|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}|$. Hence its closure is equal to the set of specializations, cf. [Hoc69, Theorem 1], and thus the set of points $x\in|\mathcal{X}^*_{\Gamma(p^\infty)}|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}|$ whose maximal generalization $\widetilde{x}\in|\mathcal{X}^*_{\Gamma(p^\infty)}(0)|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}(0)|$. Also, $\mathscr{F}\!\ell(\mathbb{Q}_p)$ is stable under generalization and specialization. Combining these results, it suffices to prove that for every maximally general point $x\in|\mathcal{X}^*_{\Gamma(p^\infty)}|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}|$, we have $x\in|\mathcal{X}^*_{\Gamma(p^\infty)}(0)|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}(0)|$ if and only if $|\pi_{\mathrm{HT}}|(x)\in\mathscr{F}\!\ell(\mathbb{Q}_p)$.

Let C be an algebraically closed complete non-Archimedean extension of $\mathbb{Q}_p$. Let $x:\mathrm{Spa}(C,\mathcal{O}_C)\to\mathcal{X}^*_{\Gamma(p^\infty)}\backslash\mathcal{Z}_{\Gamma(p^\infty)}$. By the moduli interpretation, it corresponds to a principally polarized Abelian variety A over C, with the infinite level structure $\eta:T_p(A)\to\mathbb{Z}_p^{2g}$. Let $G/\mathcal{O}_C$ be the Néron model of A/C. The point x lies in $\mathcal{X}^*_{\Gamma(p^\infty)}(0)\backslash\mathcal{Z}_{\Gamma(p^\infty)}(0)$, i.e. the Hasse invariant of A is invertible, if and only if B is ordinary, where $B/\mathcal{O}_C$ is the Abelian variety fitting into the exact sequence

$$0\to\widehat{T}\to\widehat{G}\to\widehat{B}\to0,$$

where T is a split torus over $\mathcal{O}_C$, cf. [Sch13b, Proposition 4.15], if and only if $B[p^\infty]\simeq(\mathbb{Q}_p/\mathbb{Z}_p)^g\times\mu^g_{p^\infty}$. Also, $|\pi_{\mathrm{HT}}|(x)\in\mathscr{F}\!\ell(\mathbb{Q}_p)$ if and only if $\mathrm{Lie}(\widehat{G})\otimes_{\mathcal{O}_C}C\subset T_p\widehat{G}\otimes_{\mathbb{Z}_p}C$ is a $\mathbb{Q}_p$-rational subspace, if and only if $\mathrm{Lie}(B)\otimes_{\mathcal{O}_C}C\subset T_pB\otimes_{\mathbb{Z}_p}C$ is a $\mathbb{Q}_p$-rational subspace.

Now suppose B is ordinary. Then the Hodge–Tate filtration is $\mathbb{Q}_p$-rational, as it measures the position of the canonical subgroup of level m under η. In particular, we have $|\pi_{\mathrm{HT}}|(x)\in\mathscr{F}\!\ell(\mathbb{Q}_p)$.

Conversely, suppose $|\pi_{\mathrm{HT}}|(x)\in\mathscr{F}\!\ell(\mathbb{Q}_p)$. By [SW13, Theorem 5.2.1], the p-divisible group $B[p^\infty]$ corresponds to a pair (T,W), where $T=T_p(B[p^\infty])$ is a free $\mathbb{Z}_p$-module of finite rank, and W is a C-subvector space of $T\otimes C(-1)$. The subspace W is a $\mathbb{Q}_p$-rational totally isotropic subspace as $|\pi_{\mathrm{HT}}|(x)\in\mathscr{F}\!\ell(\mathbb{Q}_p)$. Since all such subspaces are contained in one orbit under the action of $G(\mathbb{Z}_p)$, we conclude that $B[p^\infty]\simeq(\mathbb{Q}_p/\mathbb{Z}_p)^{2g}\times\mu^g_{p^\infty}$, and thus B is ordinary. □

Lemma 5.1.6. The set $\mathscr{F}\!\ell(\mathbb{Q}_p)$ is equal to the intersection of all of its open neighbourhoods in $\mathscr{F}\!\ell$.

Proof. This follows directly from the fact that $\mathscr{F}\!\ell(\mathbb{Q}_p)$ is stable under generalizations. □

Lemma 5.1.7. The quasi-compact open neighbourhoods of $\mathscr{F}\!\ell(\mathbb{Q}_p)$ are cofinal among all open neighbourhoods.

Proof. The flag variety $\mathrm{Fl}_{\mathbb{Q}_p}$ is projective, and thus $\mathscr{F}\!\ell(\mathbb{Q}_p)=\mathrm{Fl}(\mathbb{Q}_p)$ is quasi-compact, as the induced topology is the same as the p-adic analytic topology. Then we conclude by the fact that the topology of $\mathscr{F}\!\ell$ is generated by quasi-compact open subsets. □

Lemma 5.1.8. For any open subset $U\subset\mathscr{F}\!\ell$ containing a $\mathbb{Q}_p$-rational point, we have $G(\mathbb{Q}_p)\cdot U=\mathscr{F}\!\ell$.

Proof. Omitted. □

Lemma 5.1.9. Let $0 < \epsilon < 1$. There is an open subset $U \subset \mathscr{F}\ell$ containing $\mathscr{F}\ell(\mathbb{Q}_p)$ such that

$$|\pi_{\mathrm{HT}}|^{-1}(U) \subset |\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}(\epsilon)|.$$

Proof. By induction on $g \geq 0$, it sufficies to prove the similar result for the restriction $\pi = |\pi_{\mathrm{HT}}| : |\mathcal{X}_{\Gamma(p^\infty)}| \to |\mathscr{F}\ell|$ to the good reduction locus. We shall show that there exists an open subset $U \subset \mathscr{F}\ell$ containing $\mathscr{F}\ell(\mathbb{Q}_p)$ such that $\pi^{-1}(U) \subset |\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)|$, i.e. $|\mathrm{Ha}| \geq |p|^\epsilon$ on U. Consider the constructible topology $|\mathcal{X}_{\Gamma(p^\infty)}|_{\mathrm{cons}}$. It is the coarest topology such that every quasi-compact open subset becomes open and closed. Hence $\pi^{-1}(U)$ is closed in the constructible topology for every quasi-compact open neighbourhood of $\mathscr{F}\ell(\mathbb{Q}_p)$, as the map π is quasi-compact. The complement $|\mathcal{X}_{\Gamma(p^\infty)}|\backslash|\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)|$ is closed and thus quasi-compact in the constructible topology. Also, we have

$$\overline{|\mathcal{X}_{\Gamma(p^\infty)}(0)|} = \pi^{-1}(\mathscr{F}\ell(\mathbb{Q}_p)) = \bigcap_{\mathscr{F}\ell(\mathbb{Q}_p)\subset U} \pi^{-1}(U) \subset |\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)|,$$

by combining Lemma 5.1.5 and Lemma 5.1.6. Finally we obtain a desired open U satisfying $\pi^{-1}(U) \subset |\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)|$ by Lemma 5.1.7. □

Lemma 5.1.10. Let $0 < \epsilon < 1$. There are finitely many $\gamma_1, \ldots, \gamma_k \in G(\mathbb{Q}_p)$ such that

$$|\mathcal{X}^*_{\Gamma(p^\infty)}|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}| = \bigcup_{i=1}^k \gamma_i \cdot (|\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}(\epsilon)|).$$

Moreover, for such $\gamma_1, \ldots, \gamma_k$, we have

$$|\mathcal{X}^*_{\Gamma(p^\infty)}| = \bigcup_{i=1}^k \gamma_i \cdot |\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)|.$$

Proof. Let $U \subset \mathscr{F}\ell$ be an open subset containing $\mathscr{F}\ell(\mathbb{Q}_p)$ such that $|\pi_{\mathrm{HT}}|^{-1}(U) \subset |\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}(\epsilon)|$, cf. Lemma 5.1.9. Apply Lemma 5.1.8 and the fact that $\mathscr{F}\ell$ is quasi-compact, we obtain finitely many elements $\gamma_1, \ldots, \gamma_k \in G(\mathbb{Q}_p)$ such that $\mathscr{F}\ell = \cup_{i=1}^k \gamma_i \cdot U$. Taking the preimage, the first equation is clear, cf. Lemma 5.1.4.

Let V be the union $\cup_{i=1}^k \gamma_i \cdot |\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)|$. It remains to show that $V = |\mathcal{X}^*_{\Gamma(p^\infty)}|$. Note that V is a quasi-compact open subset of $|\mathcal{X}^*_{\Gamma(p^\infty)}|$, and contains $|\mathcal{X}^*_{\Gamma(p^\infty)}|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}|$. There exists $m \geq 1$ such that V is the preimage of some $V_m \subset |\mathcal{X}^*_{\Gamma(p^m)}|$, where V_m is a quasi-compact open containing $\mathcal{X}^*_{\Gamma(p^m)}\backslash\mathcal{Z}_{\Gamma(p^m)}$. It sufficies to show that $V_m = \mathcal{X}^*_{\Gamma(p^m)}$. Let $x \in \mathcal{X}^*_{\Gamma(p^m)}\backslash V_m$. Then $\{x\}$ is equal to the intersection of all open neighbourhoods of x in $\mathcal{X}^*_{\Gamma(p^m)}$. The subset V_m is quasi-compact in the constructible topology, and thus there exists an open neighbourhood U of x that is disjoint with V_m. In particular, we have $U \subset \mathcal{Z}_{\Gamma(p^m)}$, which is impossible due to dimension reason. □

Lemma 5.1.11. Let $0 \leq \epsilon < 1/2$. There exists finitely many $\gamma_1, \ldots, \gamma_k \in G(\mathbb{Z}_p)$ such that

$$|\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)| = \bigcup_{i=1}^k \gamma_i \cdot |\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a|.$$

Proof. Let $x \in |\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)|$. It corresponds to a principally polarized Abelian variety A with a level structure $\eta : T_p(A) \to \mathbb{Z}_p^{2g}$. We assume that A has good reduction. The existence of the anti-canonical subgroup $D \subset A[p]$ is equivalent to the condition that the isomorphism $A[p] \to \mathbb{F}_p^{2g}$ induced by η sends D to a subgroup disjoint with $\mathbb{F}_p^g \subset \mathbb{F}_p^{2g}$, which can always be achieved by an element in $G(\mathbb{F}_p)$. □

5.2. **The map of adic spaces.**

Definition 5.2.1. A subset $U \subset |\mathcal{X}^*_{\Gamma(p^\infty)}|$ is called affinoid perfectoid if

(1) There exists $m \geq 1$ such that U is the preimage of an affinoid open $\mathrm{Spa}(R_m, R_m^+) \subset |\mathcal{X}^*_{\Gamma(p^m)}|$.
(2) For every $m' \geq m$, if we write the preimage of $\mathrm{Spa}(R_m, R_m^+)$ in $\mathcal{X}^*_{\Gamma(p^{m'})}$ (which is necessarily affinoid) as $\mathrm{Spa}(R_{m'}, R_{m'}^+)$, then (R_∞, R_∞^+) is an affinoid perfectoid $\mathbb{Q}_p^{\mathrm{cycl}}$-algebra, where R_∞^+ is the p-adic completion of $\mathrm{colim}_{m'} R_{m'}^+$ and $R_\infty = R_\infty^+[1/p]$.

Definition 5.2.2. A subset $U \subset |\mathcal{X}^*_{\Gamma(p^\infty)}|$ is called perfectoid if it is a union of affinoid perfectoid open subsets.

Lemma 5.2.3. There exists a perfectoid space $\mathcal{X}^*_{\Gamma(p^\infty)}$ over $\mathbb{Q}_p^{\mathrm{cycl}}$ such that

$$\mathcal{X}^*_{\Gamma(p^\infty)} \sim \lim_m \mathcal{X}^*_{\Gamma(p^m)}.$$

It is covered by finitely many $G(\mathbb{Q}_p)$-translates of $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ for any $0 < \epsilon < 1/2$.

Proof. By Lemma 5.1.10, Lemma 5.1.11, and Lemma 4.4.2, we conclude that $|\mathcal{X}^*_{\Gamma(p^\infty)}|$ is perfectoid. Thus it carries the structure of a perfectoid space. The remaining two claims are both clear. □

Let $\mathcal{Z}_{\Gamma(p^\infty)} \subset \mathcal{X}^*_{\Gamma(p^\infty)}$ be the boundary.

Lemma 5.2.4. There is a unique map of adic spaces over $\mathbb{Q}_p$

$$\pi_{\mathrm{HT}} : \mathcal{X}^*_{\Gamma(p^\infty)} \backslash \mathcal{Z}_{\Gamma(p^\infty)} \to \mathscr{F}\ell$$

which realizes $|\pi_{\mathrm{HT}}|$ on topological spaces.

Proof. We repeat the arguments of Construction 5.1.3 and the result is clear. □

Definition 5.2.5. Recall the following inclusions of algebraic varieties over $\mathbb{Q}$

$$\mathrm{Fl} \to \mathrm{Gr}(2g, g) \to \mathbb{P}^{\binom{2g}{g}-1} \simeq \mathbb{P}(\wedge^g \mathbb{Q}^{2g}).$$

Let $\{e_1, \dots, e_{2g}\}$ be a basis of $\mathbb{Q}^{2g}$. Denote $e_J = e_{j_1} \wedge \cdots \wedge e_{j_g}$ for $J = \{j_1 < \cdots < j_g\} \subset \{1, \dots, 2g\}$. Then we have an affinoid open $\{|x_J| \geq 1\}$ in $\mathbb{P}(\wedge^g \mathbb{Q}^{2g})^{\mathrm{ad}}$ for every J. The preimage in $\mathscr{F}\ell$ is denoted by $\mathscr{F}\ell_J$, which is an affinoid open in $\mathscr{F}\ell$.

Lemma 5.2.6. Let $L \in \mathscr{F}\ell(\mathbb{Q}_p)$ be a totally isotropic subspace of $\mathbb{Z}_p^{2g}$. Then $L \in \mathscr{F}\ell_{\{g+1,\dots,2g\}}$ if and only if L is transverse with $\mathbb{Z}_p^g \oplus 0^g$.

Proof. Omitted. □

Lemma 5.2.7. The preimage of $\mathscr{F}\ell_{\{g+1,\dots,2g\}}(\mathbb{Q}_p)$ is given by the closure of $\mathcal{X}^*_{\Gamma(p^\infty)}(0)_a \backslash \mathcal{Z}_{\Gamma(p^\infty)}(0)_a$.

Proof. We first consider the good reduction case. It sufficies to check the rank 1 points. Let $x : \mathrm{Spa}(C, \mathcal{O}_C) \to \mathcal{X}_{\Gamma(p^\infty)}$. Suppose it corresponds to the Abelian variety $A/\mathcal{O}_C$ with level structure $\eta : T_a A \to \mathbb{Z}_p^{2g}$. We have that $\pi_{\mathrm{HT}}(x) \in \mathscr{F}\ell_{\{g+1,\dots,2g\}}(\mathbb{Q}_p)$ if and only if $x \in \mathcal{X}_{\Gamma(p^\infty)}(0)_a$, cf. Lemma 5.1.5. The result then follows from Lemma 5.2.6.

Next, we shall show that $\mathcal{X}^*_{\Gamma(p^\infty)}(0)_a \backslash \mathcal{Z}_{\Gamma(p^\infty)}(0)_a$ is mapped to $\mathcal{F}_{\{g+1,\dots,2g\}}(\mathbb{Q}_p)$. Otherwise, there exists a clopen subset of $\mathscr{F}\ell(\mathbb{Q}_p)$ disjoint from $\mathscr{F}\ell_{\{g+1,\dots,2g\}}(\mathbb{Q}_p)$ that intersects with the image of $\mathcal{X}^*_{\Gamma(p^\infty)}(0)_a \backslash \mathcal{Z}_{\Gamma(p^\infty)}(0)_a$. Taking its preimage gives a clopen subset of $\mathcal{X}^*_{\Gamma(p^\infty)}(0)_a \backslash \mathcal{Z}_{\Gamma(p^\infty)}(0)_a$ whose image in $\mathscr{F}\ell(\mathbb{Q}_p)$ is disjoint from $\mathscr{F}\ell_{\{g+1,\dots,2g\}}(\mathbb{Q}_p)$. Recall that the triple $(\mathcal{X}^*_{\Gamma(p^\infty)}(0)_a, \mathcal{Z}_{\Gamma(p^\infty)}(0)_a, \mathcal{X}_{\Gamma(p^\infty)}(0)_a)$ is good, cf. Lemma 4.4.2, which implies that any clopen subset of $\mathcal{X}^*_{\Gamma(p^\infty)}(0)_a \backslash \mathcal{Z}_{\Gamma(p^\infty)}(0)_a$ extends uniquely to a clopen subset of $\mathcal{X}^*_{\Gamma(p^\infty)}(0)_a$. Hence we obtain a nonempty clopen subset $V \subset \mathcal{X}^*_{\Gamma(p^\infty)}(0)_a$ such that

$$\pi_{\mathrm{HT}}(V \cap \mathcal{X}_{\Gamma(p^\infty)}(0)_a) = \pi_{\mathrm{HT}}(V \backslash \mathcal{Z}_{\Gamma(p^\infty)}(0)_a) \subset \mathscr{F}\ell(\mathbb{Q}_p) \backslash \mathscr{F}\ell_{\{g+1,\dots,2g\}}(\mathbb{Q}_p).$$

Intersecting with V gives a good triple

$$(V, V \cap \mathcal{Z}_{\Gamma(p^\infty)}(0)_a, V \cap \mathcal{X}_{\Gamma(p^\infty)}(0)_a).$$

In particular, the intersection $V \cap \mathcal{X}_{\Gamma(p^\infty)}(0)_a$ is nonempty. This constradicts the case of good reduction.

Finally, it suffices to prove that $\pi_{\mathrm{HT}}(x) \in \mathscr{F}\ell_{\{g+1,\dots,2g\}}(\mathbb{Q}_p)$ implies that $x \in \mathcal{X}^*_{\Gamma(p^\infty)}(0)_a \backslash \mathcal{Z}_{\Gamma(p^\infty)}(0)_a$ for every rank 1 point $x : \mathrm{Spa}(C, \mathcal{O}_C) \to \mathcal{X}^*_{\Gamma(p^\infty)} \backslash \mathcal{Z}_{\Gamma(p^\infty)}$. Choose $\gamma \in G(\mathbb{Z}_p)$ such that $\gamma \cdot x \in \mathcal{X}^*_{\Gamma(p^\infty)}(0)_a$, cf. Lemma 5.1.11, and assume that $x \notin \mathcal{X}^*_{\Gamma(p^\infty)}(0)_a \backslash \mathcal{Z}_{\Gamma(p^\infty)}(0)_a$. Then

$$\gamma \cdot x \in \mathcal{X}^*_{\Gamma(p^\infty)}(0)_a \backslash \gamma \cdot \mathcal{X}^*_{\Gamma(p^\infty)}(0)_a.$$

Repeat the argument in the previous paragraph, we obtain an element $y \in \mathcal{X}_{\Gamma(p^\infty)}(0)_a \backslash \gamma \cdot \mathcal{X}_{\Gamma(p^\infty)}(0)_a$ such that $\pi_{\mathrm{HT}}(y) \in \gamma \cdot \mathscr{F}\ell_{\{g+1,\dots,2g\}}(\mathbb{Q}_p)$. Note that $\gamma^{-1} \cdot y \in \mathcal{X}_{\Gamma(p^\infty)}(0) \backslash \mathcal{X}_{\Gamma(p^\infty)}(0)_a$, and $\pi_{\mathrm{HT}}(\gamma^{-1} \cdot y) \in \mathscr{F}\ell_{\{g+1,\dots,2g\}}(\mathbb{Q}_p)$. This constradicts the good reduction case. □

Lemma 5.2.8. For every open subset $U \subset \mathscr{F}\ell$ containing $\mathscr{F}\ell(\mathbb{Q}_p)$, there is some $\epsilon > 0$ such that

$$\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)\backslash\mathcal{Z}_{\Gamma(p^\infty)}(\epsilon) \subset \pi_{\mathrm{HT}}^{-1}(U).$$

Proof. Repeat the argument in Lemma 5.1.7. □

Lemma 5.2.9. There exists some $0 < \epsilon < 1/2$ such that

$$\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a\backslash\mathcal{Z}_{\Gamma(p^\infty)}(\epsilon)_a \subset \pi_{\mathrm{HT}}^{-1}(\mathscr{F}\ell_{\{g+1,\dots,2g\}}).$$

Proof. Choose an open neighbourhood U of $\mathscr{F}\ell(\mathbb{Q}_p)$ in $\mathscr{F}\ell$ such that $U \cap \mathscr{F}\ell_{\{g+1,\dots,2g\}}(\mathbb{Q}_p)$ is clopen in U. Let $U' = U\backslash\mathscr{F}\ell_{\{g+1,\dots,2g\}}(\mathbb{Q}_p)$. We can choose ϵ such that $\pi_{\mathrm{HT}}(\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a\backslash\mathcal{Z}_{\Gamma(p^\infty)}(\epsilon)_a) \subset U$. The preimage of U' gives a unique clopen subset $V_\epsilon \subset \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$, as the triple

$$(\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a, \mathcal{Z}_{\Gamma(p^\infty)}(\epsilon)_a, \mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a)$$

is good. The intersection of all V_ϵ for ϵ sufficiently smalle is empty, cf. Lemma 5.2.7. Since every V_ϵ is quasi-compact under the constructible topology, we have $V_\epsilon = \emptyset$ for some $\epsilon > 0$, and the proof is complete. □

Lemma 5.2.10. There exists a unique map of adic spaces

$$\pi_{\mathrm{HT}} : \mathcal{X}^*_{\Gamma(p^\infty)} \to \mathscr{F}\ell$$

extending π_{HT} on $\mathcal{X}^*_{\Gamma(p^\infty)}\backslash\mathcal{Z}_{\Gamma(p^\infty)}$.

Proof. We first show the existence. As π_{HT} is $G(\mathbb{Q}_p)$-equivariant, it sufficies to show that π_{HT} admits an extension from $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a\backslash\mathcal{Z}_{\Gamma(p^\infty)}(\epsilon)_a$ to $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ for some $\epsilon > 0$, cf. Lemma 5.1.10 and Lemma 5.1.11. It can be assumed that $\pi_{\mathrm{HT}}(\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a\backslash\mathcal{Z}_{\Gamma(p^\infty)}(\epsilon)_a) \subset \mathscr{F}\ell_{\{g+1,\dots,2g\}}$ by Lemma 5.2.9. Every bounded function on $\mathscr{F}\ell_{\{g+1,\dots,2g\}}$ pulls back to a bounded function on $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a\backslash\mathcal{Z}_{\Gamma(p^\infty)}(\epsilon)_a$, which by Riemann's Hebbarkeitssatz extends uniquely to a bounded function on $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$. Hence the extenstion exists.

For the uniqueness, it can be checked locally. Let U be an affinoid perfectoid of $\mathcal{X}^*_{\Gamma(p^\infty)}$. Let f, g be two functions on U that are equal on $U\backslash\mathcal{Z}_{\Gamma(p^\infty)}$. The subset $\{|f-g| \geq |p|^n\}$ is an open subset of U contained in the boundary, and thus must be empty. Hence $|f-g| < |p|^n$ for all n. Therefore $f = g$, and we are done. □

Appendix A. Review of deformation theory

Definition A.0.1 ([Ill71, II.1.2.1, II.1.2.3])**.** Let $A \to B$ be a map of rings. The simplicial A-algebra $P_A(B)$ is defined by $P_A(B)_0 = A[B]$ and $P_A(B)_n = A[P_A(B)_{n-1}]$ for $n \geq 1$. The standard resolution of B over A is the argumentation $P_A(B) \to B$ where B is viewed as a constant simplicial A-algebra. The cotangent complex of B over A is the simplicial B-module $L_{B/A} = \Omega^1_{P_A(B)/A} \otimes_{P_A(B)} B$.

Remark A.0.2. This definition works in a general topos.

Definition A.0.3 ([Ill72, VII.1.1.1])**.** Let S be a scheme. Let S_{zar} be the small Zariski site over S. Let S_{fpqc} be the big fqpc site over S. The natural inclusion $S_{\mathrm{zar}} \to S_{\mathrm{fpqc}}$ induces a geometric map $(\epsilon^*, \epsilon_*) : \mathrm{Sh}(S_{\mathrm{zar}}) \rightleftarrows \mathrm{Sh}(S_{\mathrm{fpqc}})$.

Definition A.0.4. Let $f : X \to Y$ be a map of schemes. The cotangent complex is $L_{X/Y}$.

Definition A.0.5. Let S be a scheme. Let G be a group scheme over S that is flat and locally of finite presentation. Let $e : S \to G$ be the unit. The co-Lie complex is $\ell_G = Le^*L_{G/S}$, and the Lie complex is $\ell_G^\vee = R\underline{\mathrm{Hom}}(\ell_G, \mathcal{O}_S)$. Define $\underline{\ell}_G = L\epsilon^*\ell_G$.

Remark A.0.6. A group scheme G flat over S is always a local complete intersection over S. Then the cotangent complex $L_{G/S}$ has perfect amplitude in $[-1, 0]$, and thus ℓ_G has perfect amplitude in $[-1, 0]$, $\ell_G^\vee$ has perfect amplitude in $[0, 1]$.

If G is smooth over S, then $L_{G/S} = \Omega_{G/S}$ is locally free, and $\ell_G^\vee$ coincides with the Lie algebra $\mathrm{Lie}(G)$ of G.

In particular, if $0 \to H \to A \to B \to 0$ is a short exact sequence of commutative group schemes over S, with H finite locally free, and A, B smooth, then $\ell_H^\vee$ is represented be the two-term complex $\mathrm{Lie}(A) \to \mathrm{Lie}(B)$.

Lemma A.0.7 ([Ill72, Theorem VII.4.2.5])**.** Let $f : S \to T$ be a map of schemes. Let $i : S \to S'$ be a T-extension by a quasi-coherent module I. Let A be a "schéma en anneaux" over T that is, as a scheme over T, tor-independent (c.f. [FJJ+71, Definition III.1.5]) with both S and S'. Let F (resp. G') be "schéma en A-modules" that are flat and locally of finite presentation over S (resp. S'). Let G be a "schéma en A-module" over S induced by G'. Let $u : F \to G$ be a morphism of "schémas en A-modules". Let K be the complex fitting into the distinguished triangle $K \to \ell_F^\vee \to \ell_G^\vee \to K[1]$. It is an object in $D(A \otimes_{\mathbb{Z}}^L \mathcal{O})$. Then there is an obstruction $\omega(u, G') \in \mathrm{Ext}^2_A(F, K \otimes_{\mathcal{O}}^L \epsilon^* I)$ which is zero if and only if there exists a pair (F', u') where F' is a deformation of F as "un schéma en A-modules" flat over S' and a map $u' : F' \to G'$ extending u.

Lemma A.0.8. Let S be a scheme. Let $i : S \to S'$ be an extension by a quasi-coherent module I. Suppose S and S' are both tor-indepeendent with $\mathrm{Spec}(\mathbb{Z})$. Let F (resp. G') be commutative group schemes over S (resp. S') that are flat and locally of finite presentation. Let G be a commutative group scheme over S induced by G'. Let $u : F \to G$ be a morphism of group schemes over S. Let K be the cone of the map $\ell_F^\vee \to \ell_G^\vee$. There is an obstruction $\omega(u, G') \in \mathrm{Ext}^1(F, K \otimes^L I)$ which vanishes if and only if there exists a pair (F', u') where F' is a deformation of F as a commutative group scheme that is flat over S', and $u' : F' \to G'$ is a map extending u.

Lemma A.0.9 ([Sch15, Theorem III.2.1])**.** Let A be a ring. Let G and H be commutative group schemes over A that are flat and of finite presentation, with a group map $u : H \to G$. Let $B \to A$ be a square-zero thickening with the argumentation ideal J. Let $\widetilde{G}$ be a lift of G to B. Let K be a cone of the map $\ell_H^\vee \to \ell_G^\vee$ of Lie complexes. Then there is an obstruction class $\omega \in \mathrm{Ext}^1(H, K \otimes^L J)$ which vanishes if and only if there exists a pair $(\widetilde{H}, \widetilde{u})$ where $\widetilde{H}$ is a flat commutative group scheme over B, and $\widetilde{u} : \widetilde{H} \to \widetilde{G}$ is a map lifting $u : H \to G$. Moreover, the obstruction class is functorial in J, in the following sense. If $B' \to A$ is another square-zero thickening with the argumentation ideal J', with a map $B \to B'$ over A, then $\omega' \in \mathrm{Ext}^1(H, K \otimes^L J')$ is the image of $\omega \in \mathrm{Ext}^1(H, K \otimes^L J)$ under the map $J \to J'$.

Appendix B. Review of perfectoid spaces

Definition B.0.1. Let $\mathfrak{Y}$ be a flat t-adic formal scheme over $\mathbb{F}_p[[t^{1/(p-1)p^\infty}]]$. Let $\Phi : \mathfrak{Y} \to \mathfrak{Y}$ be the ralative Frobenius. Then the inverse limit $\lim_\Phi \mathfrak{Y}$ is representable by a perfect flat t-adic formal scheme $\mathfrak{Y}^{\mathrm{perf}}$ over $\mathbb{F}_p[[t^{1/(p-1)p^\infty}]]$, called the perfection of $\mathfrak{Y}$.

Locally,

$$(\mathrm{Spf}(S))^{\mathrm{perf}} = \mathrm{Spf}(S^{\mathrm{perf}})$$

where S^{perf} is the t-adic completion of $\lim_\Phi S$.

Definition B.0.2. Let $\mathcal{Y}$ be an adic space over $\mathbb{F}_p((t^{1/(p-1)p^\infty}))$. Let $\Phi : \mathcal{Y} \to \mathcal{Y}$ be the ralative Frobenius. Then there exists a unique perfectoid space $\mathcal{Y}^{\mathrm{perf}}$ over $\mathbb{F}_p((t^{1/(p-1)p^\infty}))$, called the perfection of $\mathcal{Y}$, such that

$$\mathcal{Y}^{\mathrm{perf}} \sim \lim_\Phi \mathcal{Y}.$$

Locally,

$$\mathrm{Spa}(S, S^+)^{\mathrm{perf}} = \mathrm{Spa}(S^{\mathrm{perf}}, S^{\mathrm{perf},+})$$

where $S^{\mathrm{perf},+}$ is the t-adic completion of $\lim_\Phi S^+$, and $S^{\mathrm{perf}} = S^{\mathrm{perf},+}[1/t]$.

Lemma B.0.3. Let K be a perfectoid field. Let $\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2$ be perfectoid spaces over K, with finite étale maps $\mathcal{Y}_1 \to \mathcal{X}$ and $\mathcal{Y}_2 \to \mathcal{X}$. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map over $\mathcal{X}$. Let $\mathcal{U} \subset \mathcal{X}$ be an open subset such that the restriction map $H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}) \to H^0(\mathcal{U}, \mathcal{O}_\mathcal{U})$ is injective. If $f|_\mathcal{U}$ is an isomorphism, then f is an isomorphism.

Proof. The points of $\mathcal{X}$ where the map of stalks induced by f is an isomorphism is open and closed. If f is not an isomorphism, then there exists a non-trivial idempotent $e \in H^0(\mathcal{X}, \mathcal{O}_\mathcal{X})$ such that $e|_\mathcal{U} = 1$. However this contradicts the condition that the map $H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}) \to H^0(\mathcal{U}, \mathcal{O}_\mathcal{U})$ is injective. □

Lemma B.0.4. Let (R, R^+) be a Tate Huber pair. There is a bijection of sets between $|\mathrm{Spa}(R, R^+)|$ and the set $\{(L, L^+, \phi)\}/\sim$, where (L, L^+) is a non-discrete affinoid field and $\phi : (R, R^+) \to (L, L^+)$ is a map of Huber pairs such that $\phi(R) \subset L$ is dense.

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