Prismatic Cohomology

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July 6, 2025

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1 Introduction

Prismatic cohomology is a new cohomology theory for p-adic formal schemes, introduced in the ground-breaking work [BS22]. It unifies varies p-adic cohomology theories, including cyrstalline, de Rham, étale, and q-de Rham cohomology (also developed in [BS22]).

The fundamental objects in prismatic cohmology are prisms, which plays a role analogous to that of a pd-algebra in crystalline cohomology. A prism is a pair (A, I) where A is ring equipped with a Frobenius lift (in the derived sense), and I is an ideal satisfying some technical conditions, such that A is (derived) (p, I)-complete.

Moreover, prisms generalize the notion of perfectoid rings defined in [BMS18], corresponding to a "de-perfected" perfectoid ring in the following sense.

Theorem 1.1. The category of "perfect" prisms is equivalent to the category of (integral) perfectoid rings.

For technical reasons, we restrict to the class of bounded prisms (see Definition 3.8). Let (A, I) be a bounded prism. For every (smooth) p-adic formal scheme X over A/I, we shall associate the prismatic site $(X/A)_{\Delta}$ to X, and thus the prismatic cohomology $\Delta_{X/A}$, defined in a way similar to crystalline cohomology. The object $\Delta_{X/A}$ recovers many p-adic cohomology theories. Here (and throughout this note) we state the results in the affine case for simplicity. The global results then follow by formal reasons.

Theorem 1.2. Let (A, I) be a bounded prism. Let R be a p-completely smooth A/I-algebra.

(1) Assume that (A, I) is crystalline. The crystalline comparison is

$$\Delta_{R/A} \widehat{\otimes}^L_{A,\phi} A \simeq R\Gamma_{\mathrm{crys}}(R/A).$$

(2) The Hodge–Tate comparison is

$$H^\bullet(\Delta_{R/A} \otimes^L_A A/I)\{\bullet\} \simeq \Omega^\bullet_{R/(A/I)}.$$

(3) Assume that (A, I) is perfect and $I = (d)$. The étale comparison is

$$R\Gamma(\mathrm{Spec}(R[1/p]), \mathbb{Z}/p^n) \simeq (\Delta_{R/A}[1/d]/p^n)^{\phi=1}.$$

We do not discuss the de Rham comparison, the q-de Rham comparison, and the relation with homotopy theory in this note.

1.1 Notations

Some of the notations and conventions are summarized below.

- Let p be a prime.
- For an A/I-module M, we write $M\{i\} = M \otimes_{A/I} I^i/I^{i+1}$.
- For a ring A, we write $D(A)$ for the derived ∞-category of A.
- We use "$\bullet$" to denote index of complexes, (co-)simplicial objects, and dgas. It is suppressed for objects in the derived category.

2 Preliminaries

2.1 Animated rings

Definition 2.1. Let R be a ring. Let Poly_R be the category of polynomial rings over R in finitely many variables. Note that the category Poly_R admits finite coproducts given by the tensor product over R.

Lemma 2.2. Let R be a ring. The following categories are equivalent.

(1) The category of R-algebras.

(2) The full subcategory of $\mathrm{Fun}(\mathrm{Poly}_R^{\mathrm{op}}, \mathrm{Set})$ spanned by functors preserving finite products.

Definition 2.3. Let R be a ring. An animated R-algebra is a functor $\mathrm{Poly}_R^{\mathrm{op}} \to \mathrm{Ani}$ preserving finite products. The ∞-category AniAlg_R of animated R-algebras is the full subcategory of $\mathrm{Fun}(\mathrm{Poly}_R^{\mathrm{op}}, \mathrm{Ani})$.

Remark 2.4. Let R be a ring. A simplicial R-algebra is a simplicial object in the category of R-algebras. Let sAlg_R be the category of simplicial R-algebras. It is equivalent to the full subcategory of $\mathrm{Fun}(\mathrm{Poly}_R^{\mathrm{op}}, \mathrm{sSet})$ spanned by functors preserving finite products. Hence the category sAlg_R is equipped with a simplicial model structure induced by the Quillen model structure on sSet, see [Lur09, Proposition 5.5.9.1]. Moreover, the ∞-category associated to sAlg_R is equivalent to AniAlg_R by [Lur09, Corollary 5.5.9.3].

Remark 2.5. Alternatively, we can apply the process of animating a cocomplete category generated under colimits by the full-category of objects strongly of finite presentation, see [ČS24, Section 5.1.1]. For a ring R, the following ∞-categories are eqvuivalent.

(1) AniAlg_R in Definition 2.3.

(2) $\mathrm{Ani}(\mathrm{Alg}_R)$, the animation of the cateegory of R-algebras.

(3) $\mathrm{Ani}(\mathrm{Ring})_{R/}$, the slice category under the (static) ring R.

For the equivalence between (2) and (3), see [Lurb, Corollary 25.1.4.4].

Lemma 2.6. Let R be a ring. Let $\mathcal{C}$ be an ∞-category that admits (small) sifted colimits. Let $\mathrm{Fun}_\Sigma(\mathrm{AniAlg}_R, \mathcal{C})$ be the full subcategory of $\mathrm{Fun}(\mathrm{AniAlg}_R, \mathcal{C})$ spanned by functors preserving (small) sifted colimits. Then composition with the embedding $\mathrm{Poly}_R \to \mathrm{AniAlg}_R$ induces an equivalence

$$\mathrm{Fun}_\Sigma(\mathrm{AniAlg}_R, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Poly}_R p, \mathcal{C})$$

of ∞-categories.

Proof. See [Lurb, Proposition 25.1.1.5]. □

Definition 2.7. Let R be a ring. There is an essentially unique functor θ from AniAlg_R to the ∞-category of connective $\mathbb{E}_\infty$-R-algebras, extending the identity on Poly_R, by Lemma 2.6. For an animated R-algebra A, the connective $\mathbb{E}_\infty$-R-algebra $\theta(A)$ is called the underlying $\mathbb{E}_\infty$-R-algebra of A and denoted by A.

Definition 2.8. Let A be an animated ring. The ∞-category $\mathrm{Mod}(A)$ of A-modules is defined as the ∞-category of modules over the underlying $\mathbb{E}_\infty$-ring of A. The ∞-category of animated A-modules is the full subcategory of $\mathrm{Mod}(A)$ of connective objects. Alternatively, animated modules can be obtained via animating the category of pairs (R, M) where R is a ring and M is an R-module.

Definition 2.9. By animating the functor $(A, M) \mapsto \mathrm{Sym}_A(M)$, we obtain the (derived) symmetric algebra for animated modules. Let A be an animated ring. Let M be an animated A-module. Let $f : M \to A$ be a map of animated A-modules. The quotient ring $A /\!\!/ M$ is defined via the pushout diagram

$$\begin{array}{ccc} \mathrm{Sym}_A(M) & \xrightarrow{\ 0\ } & A \\ \big\downarrow{\scriptstyle f} & & \big\downarrow \\ A & \longrightarrow & A /\!\!/ M. \end{array}$$

We define quotient modules in a similar fashion.

Remark 2.10. Let A be a ring. Let $f_1, \ldots, f_r$ be elements of A. Then $A /\!\!/ (f_1, \ldots, f_r)$ can be represented by the Koszul complex $\mathrm{Kos}(A; (f_1, \ldots, f_r))$.

Definition 2.11. Let A be an animated ring. Let M be an animated A-module. We say that M is flat (resp. faithfully flat) over A if $\pi_0(M)$ is a flat (resp. faithfully flat) $\pi_0(A)$-module and if the natural morphism $\pi_i(A) \otimes_{\pi_0(A)} \pi_0(M) \to pi_i(M)$ is an isomorphism for every i.

2.2 Derived completion

Definition 2.12. Let A be an animated ring. Let $I = (f_1, \ldots, f_r) \subset \pi_0(A)$ be a finitely generated ideal. Let M be an A-module. The (derived) I-completion of M is defined as

$$\widehat{M} = \lim_{n \geq 0} M /\!\!/ I^n.$$

We say that M is (derived) I-complete if the natural map $M \to \widehat{M}$ is an isomorphism. The full subcategory of I-complete A-modules is usually denoted by $\widehat{D}(A)$.

Lemma 2.13. Let A be an animated ring. Let $I = (f_1, \dots, f_r) \subset \pi_0(A)$ be a finitely generated ideal. The inclusion functor $\widehat{D}(A) \to D(A)$ admits a left adjoint given by the I-completion functor.

Proof. See [Sta, Tag 091V] for the case A is discrete. □

Lemma 2.14. Let A be a ring. Let I be a finitely generated ideal of A. Let $K \in D(A)$ be I-complete. If $K \otimes_A^L A/I = 0$, then $K = 0$.

Proof. See [Sta, Tag 0G1U]. □

Definition 2.15. Let A be an animated ring. Let $I = (f_1, \dots, f_n) \subset \pi_0(A)$ be a finitely generated ideal. Let B be an I-complete animated A-algebra. A sequence $x_1, \dots, x_r \in \pi_0(B)$ is called I-completely regular relative to A if the map $A /\!\!/ (f_1, \dots, f_n) \to B /\!\!/ (f_1, \dots, f_n, x_1, \dots, x_r)$ of animated rings is flat.

Definition 2.16. Let A be a ring. Let I be a finitely generated ideal of A.

(1) Let M be an A-module where A is viewed as an animated ring. We say that M is I-completely flat if $M /\!\!/ I$ is discrete and flat over A/I.

(2) A ring map $A \to B$ is called I-completely smooth (resp. étale) if $B /\!\!/ I$ is discrete and smooth (resp. étale) over A/I.

Lemma 2.17. Let A be a ring. Let I be a finitely generated ideal of A. Let M be a flat (resp. faithfully flat) A-module. Then the I-completion $\widehat{M}$ is I-completely flat (resp. faithfully flat) over A.

Lemma 2.18. Let R be a ring. Let I be a finitely generated ideal of R. Let $R \to S$ be a ring map of finite I-complete tor-amplitude. Then the I-completed base-change functor $- \widehat{\otimes}_R^L S$ commutes with totalization of cosimplicial algebras (i.e. taking limit in the derived category).

Proof. This is [BS22, Lemma 4.22]. □

2.3 Čech cohomological descent

Definition 2.19. Let C be a category. An object $X \in C$ is called weakly final if every object of C admits a morphism to X.

Lemma 2.20. Let C be a site. Suppose h_X is a sheaf represented by a weakly final object X of C. Then the map $h_X \to *$ of sheaves is surjective.

Lemma 2.21. Let C be a topos. Let $Y \to X$ be a surjective morphism in the topos C. Let $Y_\bullet$ be the Čech nerve of $Y \to X$. Let A be an abelian group object of C. Then $R\Gamma(X, A)$ is computed by the cosimplicial object $R\Gamma(Y_\bullet, A)$.

Proof. Since C is a topos, the map $Y \to X$ is an effective epimorphism, and thus $Y_\bullet \to X$ is a hypercovering. The result then follows from cohomological descent, see [Sta, Tag 09VX]. □

2.4 Cosimplicial computations

Remark 2.22. Let M be a commutative additive monoid. We have a cosimplicial object EM defined as follows.

(1) The simplices are $[n] \mapsto M^{\oplus(n+1)}$.

(2) The face map $d_i : M^{\oplus n} \to M^{\oplus(n+1)}$ is the inclusion that maps into all the components except the i-th one.

(3) The degeneracy map $s_i : M^{\oplus(n+1)} \to M^{\oplus n}$ is reducing the i-th and $(i+1)$-th components using the monoid operation. The cosimplicial object EM can be viewed as the Čech nerve of $0 \to M$. Moreover, the construction $M \mapsto EM$ is functorial in M.

Remark 2.23. Let A be a $\mathbb{F}_p$-algebra. Let $B = A[X_1, \ldots, X_r]$. Then B can be identified with $A[M]$ for the monoid $M = \mathbb{N}^r$. Moreover, the relative Frobenius $B^{(1)} \to B$ identifies with the map $A[pM] \to A[M]$ induced by the inclusion $pM \to M$.

Remark 2.24. Let A be a $\mathbb{F}_p$-algebra. Let B a polynomial A-algebra of finitely many variables. Let $B^\bullet$ be the Čech nerve of $A \to B$. Let $B^{\bullet,(1)}$ be the (termwise) Frobenius twist of $B^\bullet$ and in particular we have the relative Frobenius $B^{\bullet,(1)} \to B^\bullet$ over A.

Write $B = A[M]$ with $M = \mathbb{N}^r$ for $r \geq 0$. The map $B^{\bullet,(1)} \to B^\bullet$ can be identified with $A[E(pM)] \to A[EM]$ induced by the inclusion $pM \to M$. Let $S = \{0, \ldots, p-1\}^r \subset M$. We have a semi-cosimplicial set ES defined in a similar way as EM, i.e. $(ES)^n = S^{n+1}$, and the face map $d_i : (ES)^{n-1} \to (ES)^n$ is the inclusion to all components except the i-th one and zero for the i-the component of $(ES)^n$. In particular, we have an inclusion $ES \subset EM$ of semi-cosimplicial sets. The decomposition $M = \sqcup_{s \in S} s + pM$ can be upgraded to

$$M^{\oplus(n+1)} = \bigsqcup_{s \in S^{n+1}} s + pM^{\oplus(n+1)}.$$

It follows that the semi-cosimplicial A-algebra $B^\bullet \simeq A[EM]$ is a free module over the semi-cosimplicial A-algebra $B^{\bullet,(1)} \simeq A[E(pM)]$, and

$$A[E(pM)] \otimes_A A[ES] \simeq A[EM].$$

Lemma 2.25. Let A be a $\mathbb{F}_p$-algebra. Let B a polynomial A-algebra. Let $B^\bullet$ be the Čech nerve of $A \to B$. Let $B^{\bullet,(1)}$ be the (termwise) Frobenius twist of $B^\bullet$. Let $N^\bullet$ be a cosimplicial $B^{\bullet,(1)}$-module. Then the natural map

$$N^\bullet \to N^\bullet \otimes_{B^{\bullet,(1)}} B^\bullet$$

gives a quasi-isomorphism on associated (unnormalized) chain complexes of A-modules.

Proof. Assume that B is a polynomial algebra in finitely many variables. We have

$$\begin{aligned} N^\bullet \otimes_{B^{\bullet,(1)}} B^\bullet &\simeq N^\bullet \otimes_{A[E(pM)]} A[EM] \\ &\simeq N^\bullet \otimes_{A[E(pM)]} A[E(pM)] \otimes_A A[ES] \\ &\simeq N^\bullet \otimes_A A[ES] \end{aligned}$$

as semi-cosimplicial A-modules. It remains to check that the map $A \to A[ES]$ of semi-cosimplicial A-modules is a homotopy equivalence, and this is clear as it can be viewed as the Čech nerve of $A \to A[S]$ which clearly admits a section. ☐

2.5 Crystalline cohomology

Remark 2.26. Let A be a p-torsion-free $\mathbb{Z}_{(p)}$-algebra. Let R be a smooth A/p-algebra. Let $P \to R$ be a surjection of A-algebras where P is an ind-smooth A-algebra. Let J be the kernel of $P \to R$. Then D, the p-completion of the subring of $P[1/p]$ generated by P and $\{\gamma_n(x); x \in J, n \geq 1\}$, is the p-completed pd-envelope of $P \to R$. In particular, $D \to R$ is a surjection of A-algebras with p nilpotent on D.

Construction 2.27. Let A be a ring. Let R be a smooth A/p-algebra. Let $B_0 \to R$ be a surjection of A-algebras with B_0 being the p-completion of a polynomial A-algebra. Let J_0 be the kernel of $B_0 \to R$. Let B be the p-completion of the free δ-A-algebra on B_0. Let $R' = B/J$ where $J = J_0 B$. Let D_0 (resp. D) be the p-completed pd-envelope of $J_0 \subset B_0$ (resp. $J \subset B$). The situation is summarized in the diagram

$$\begin{array}{ccccc} A & \longrightarrow & B_0 & \longrightarrow & B \\ \big\downarrow & & \downarrow & & \downarrow \\ \big\downarrow & & D_0 & \longrightarrow & D \\ \big\downarrow & & \downarrow & & \downarrow \\ A/p & \longrightarrow & R & \longrightarrow & R'. \end{array}$$

For every $n \geq 1$, the rings D_0/p^n (resp. D/p^n) identifies with the pd-envelope of $B_0/p^n \to R$ (resp. $B/p^n \to R'$). In particular, we obtain an object $(D/p^n \to R' \leftarrow R)$ of the big crystalline site $\mathrm{CRYS}(R/A)$. The pro-object $(D/p^n \to R' \leftarrow R)_{n\geq 1}$ of $\mathrm{CRYS}(R/A)$ is weakly initial, i.e. every object of $\mathrm{CRYS}(R/A)$ admits a morphism from some term of the pro-object (note that this is equivalent to being a weakly initial object in the category $\mathrm{Pro}(\mathrm{CRYS}(R/A))$ of pro-objects). Let $D^\bullet$ be the cosimplicial object obtained by applying the construction $B_0 \mapsto D$ to the Čech nerve of $A \to B_0$. Then $(D^\bullet/p^n \to (R')^\bullet \leftarrow R)_{n\geq 1}$ is the Čech nerve of the pro-object. Therefore $R\Gamma_{\mathrm{crys}}(R/A)$ is computed by $D^\bullet$, cf. [BdJ11, Lemma 2.4] and [Sta, Tag 07LH].

2.6 δ-rings

Definition 2.28. A δ-ring is a $\mathbb{Z}_{(p)}$-algebra A equipped with a map $\delta : A \to A$ of sets such that

(1) $\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$;

(2) $\delta(x + y) = \delta(x) + \delta(y) + \frac{1}{p}(x^p + y^p - (x + y)^p)$.

An element d of a δ-ring A is called distinguished if $\delta(d)$ is a unit of A.

Definition 2.29. Let (A, δ) be a δ-ring. The associated Frobenius endomorphism is $\phi_A : A \to A$ defined by

$$\phi_A(x) = x^p + p\delta(x).$$

It is a lift of the Frobenius on A/p.

Remark 2.30. Let $W_2(R)$ be the ring of Witt vectors of length 2. Recall that addition and multiplication on $W_2(R)$ are defined as

$$(x_0, y_0) + (x_1, y_1) = \Big(x_0 + x_1, y_0 + y_1 + \frac{1}{p}\big[x_0^p + x_1^p - (x_0 + x_1)^p\big]\Big)$$

and

$$(x_0, y_0) \cdot (x_1, y_1) = (x_0x_1, x_0^p y_1 + x_1^p y_0 + py_0y_1).$$

Thus it is clear that δ-structures on R are equivalent to sections of the natural map $W_2(R) \to R$.

Remark 2.31. The functor $W_2(-)$ can be extended to the category of animated rings using Lemma 2.6. We then define an animated δ-ring as an animated $\mathbb{Z}_{(p)}$-algebra R equipped with a section of $W_2(R) \to R$, cf. [BL22, Definition A.11]. The resulting category of animated δ-ring is equivalent to the category obtained via animating the category of δ-rings, cf. [Mao21, Definition 5.4].

Lemma 2.32. The category of δ-rings admits all limits and colimits, perserved by the forgetful functor to sets.

Proof. The δ-structure on limits can be constructed directly. For colimits, we can use the characterization with W_2. ☐

Remark 2.33. By formal reasons, the forgetful from δ-rings to sets admits a left adjoint, that is, "adjoining elements", denoted by $A\{\cdot\}$. However, in the following, we mostly adjoin elements in the ∞-category of animated δ-rings (say, via a pushout square), and then show the resulting animated δ-ring is discrete.

Lemma 2.34. Let A be a δ-ring. Let I be a finitely generated ideal of A containing p. Then the δ-structure on A can be extended uniquely to a δ-structure on the I-completion of A.

Proof. See [BS22, Lemma 2.18]. ☐

Lemma 2.35. Let A be a δ-ring. Let d be an element of A. Assume that $p, d \in \mathrm{rad}(A)$. Then d is an distinguished element if and only if $p \in (d, \phi(d))$.

Proof. See [BS22, Lemma 2.25]. ☐

Lemma 2.36. Let A be a δ-ring. Let $d = fh$ be an distinguished element of A. If $f, p \in \mathrm{rad}(A)$, then f is distinguished in A and h is a unit of A.

Proof. We have

$$\delta(d) = f^p \delta(h) + h^p \delta(f) + p\delta(f)\delta(h).$$

The claim follows directly. □

Lemma 2.37. Let $\mathbb{Z}_{(p)}\{x\}$ be the free δ-ring on $\{x\}$.

(1) The ring $\mathbb{Z}_{(p)}\{x\}$ is the polynomial $\mathbb{Z}_{(p)}$-algebra on $\{x, \delta(x), \delta^2(x), \dots\}$.

(2) The associated Frobenius ϕ on $\mathbb{Z}_{(p)}$ is faithfully flat.

Proof. Proof of (1). Let A be the polynomial $\mathbb{Z}_{(p)}$-algebra $\mathbb{Z}_{(p)}[x_0, x_1, \dots]$. We define an endomorphism ϕ of A by

$$\phi(x_i) = x_i^p + p x_{i+1}$$

for $i \geq 0$. Then ϕ lifts the Frobenius on A/p, and thus it corresponds to a unique δ-structure on A as A is p-torsion-free. Unwinding the definitions, wee see that $\delta(x_i) = x_{i+1}$. The universal property is clear.

Proof of (2). We first write $\phi : A \to A$ as the filtered colimit of

$$\phi_i : \mathbb{Z}_{(p)}[x_0, \dots, x_i] \to \mathbb{Z}_{(p)}[x_0, \dots, x_{i+1}].$$

It suffices to show that both $\phi_i[1/p]$ and ϕ_i/p are faithfully flat. It is clear that $\phi_i[1/p]$ can be written as

$$\phi_i[1/p] : \mathbb{Q}[x_0, \phi(x_0), \dots, \phi^(i-1)(x_0)] \to \mathbb{Q}[x_0, \phi(x_0), \dots, \phi^i(x_0)]$$

where $\phi_i[1/p]$ shifts generators. Hence $\phi_i[1/p]$ is faithfully flat. It is also clear that ϕ/p can be identified with the inclusion

$$\phi_i/p : \mathbb{F}_p[x_0, \dots, x_i] \to \mathbb{F}_p[x_0, \dots, x_{i+1}]$$

which is clearly faithfully flat. □

Lemma 2.38. Let A be a p-complete animated δ-ring. Let B be a p-complete animated δ-A-algebra. Let $x_1, \dots, x_r$ be a sequence in $\pi_0(B)$ that is p-completely regular relative to A. Then $C = B\{x_1/p, \dots, x_r/p\}^\wedge$ is p-completely flat over A.

Proof. See [BS22, Corollary 2.44]. □

Lemma 2.39. Let A be a p-torsion-free δ-ring. Let $f_1, \dots, f_r$ be elements of A such that the images in A/p form a regular sequence. Let $A\{\phi(f_1)/p, \dots, \phi(f_r)/p\}$ be the animated δ-ring obtained by freely adjoining (in the category of animated δ-rings) the elements $\phi(f_i)/p$ to A. Then $A\{\phi(f_1)/p, \dots, \phi(f_r)/p\}$ is discrete, p-torsion-free, and identifies with $D_{(f_1,\dots,f_r)}(A)$, the pd-envelope of $(f_1, \dots, f_r) \subset A$.

Proof. See [BS22, Corollary 2.39]. □

Lemma 2.40. Let A be a δ-ring. Let I be a finitely generated ideal of A containing p. Let B be an I-complete I-completely étale A-algebra. Then B admits a unique δ-structure compatible with A.

Proof. This is [BS22, Lemma 2.18]. We only remark that the proof relies on the fact (a version of Elkik's algebraization) that an I-completely étale A-algebra can be written as the I-completion of some étale A-algebra, see [BS22, Footnote 6]. □

2.7 Perfectoid rings

Definition 2.41 ([BMS18, Definition 3.5]). A ring R is perfectoid if it is p-adically complete, there is some $\pi \in R$ such that $\pi^p = pu$ for some $u \in R^\times$, the Frobenius $x \mapsto x^p$ on R/p is surjective, and the kernel of Fontaine's map $\theta : A_{\mathrm{inf}}(R) \to R$ is principal.

Lemma 2.42 ([BMS19, Proposition 4.19]). Let R be a perfectoid rings.

(1) The kernel of $\theta : A_{\mathrm{inf}}(R) \to R$ is generated by a non-zero-divisor ξ of the form $p + [\pi^\flat]^p \alpha$ where $\pi^\flat = (\pi, \pi^{1/p}, \dots) \in R^\flat$ and $\alpha \in A_{\mathrm{inf}}(R)$.

(2) We have $R[p^\infty] = R[p]$. In particular, the ring R has bounded p^∞-torsion.

2.8 The arc topology

Remark 2.43. Recall that the valuation group of a valuation ring of rank 1 can be embedded in the the additive group $\mathbb{R}$. We say that a valuation ring is eudoxian if its rank is ≤ 1, following [Ked21, Definition 20.2.1].

Definition 2.44 ([BM21, Definition 1.2])**.** A map $f : Y \to X$ of (qcqs) schemes is called an arc cover if for every eudoxian valuation ring V and a map $\mathrm{Spec}(V) \to X$, there is an extension $V \to W$ of eudoxian valuation rings and a map $\mathrm{Spec}(W) \to Y$ lifting f. The arc topology on the category of schemes is the Grothendieck topology where $\{f_i : Y_i \to X\}_{i \in I}$ is a covering family if for every affine open $V \subset X$, there exist a map $t : K \to I$ of sets with K finite, and affine opens $U_k \subset f_{t(k)}^{-1}(V)$ for $k \in K$ such that the induced map $\sqcup_k U_k \to V$ is an arc cover.

The following variant is used in the étale comparison theorem.

Definition 2.45. A map $f : Y \to X$ of schemes is called an arc-p cover if it satisfies the condition for an arc-cover in Definition 2.44 for every p-complete eudoxian valuation ring V. The arc-p topology is defined similarly using arc-p covers.

Lemma 2.46. Let R be a $\mathbb{Z}_{(p)}$-algebra. Let $\mathcal{F}$ be a torsion sheaf on $\mathrm{Spec}(R)_{\text{ét}}$. Then the functor

$$S \mapsto R\Gamma(\mathrm{Spec}(S_p^{\wedge}[1/p]), \mathcal{F})$$

is a sheaf for the arc-p topology.

Proof. See [BM21, Corollary 6.17]. □

3 Prisms

3.1 Basics

Definition 3.1. A δ-pair is a pair (A, I) consisting of a δ-ring A and an ideal I of A.

Definition 3.2. A prism is a δ-pair (A, I) satisfying the following properties.

(1) The ideal I defines a Cartier divisor on $\mathrm{Spec}(A)$, i.e. I is an invertible A-module.

(2) The ring A is (p, I)-complete.

(3) We have $p \in I + \phi(I)A$.

Remark 3.3. The first condition implies that the ideal I is finitely generated. The third condition is satisfied when I is generated by a distinguished element, cf. Lemma 2.35.

Lemma 3.4. Let (A, I) be a prism. Then there exists a faithfully flat map $A \to A'$ of δ-rings such that IA' is generated by a distinguished element d.

Proof. This is [BS22, Lemma 3.1]. □

Lemma 3.5. Let $(A, I) \to (B, J)$ be a map of prisms. Then we have a natural isomorphism $I \otimes_A B \simeq J$ of B-modules. In particular, we have $J = IB$.

Proof. This is [BS22, Lemma 3.5]. □

Lemma 3.6. Let (A, I) be a prism. Let $A \to B$ be a map of δ-rings. Assume that B is (p, I)-complete as an A-algebra. Then (B, IB) is a prism if and only if $B[I] = 0$.

Proof. This is [BS22, Lemma 3.5]. □

Definition 3.7. A map $(A, I) \to (B, J)$ of prisms is flat (resp. faithfully flat) if the underlying ring map $A \to B$ is (p, I)-completely flat (resp. faithfully flat).

3.2 Bounded prisms

Definition 3.8. Let (A, I) be a prism.

(1) It is called bounded if A/I has bounded p^∞-torsion.

(2) It is called orientable if the ideal I is principal, and a choice of generator is called an orientation. It is called oriented if an orientation is chosen.

Lemma 3.9. Let (A, I) be a bounded prism.

(1) The ring A is classically (p, I)-complete.

(2) Let $M \in D(A)$ be (p, I)-complete and (p, I)-completely flat. Then M is discrete and classically (p, I)-complete. Moreover, we have $M[I] = 0$ and M/IM has bounded p^∞-torsion.

(3) There exists a faithfully flat map $(A, I) \to (B, IB)$ of prisms where (B, IB) is bounded and orientable.

Proof. This is [BS22, Lemma 3.7]. □

Remark 3.10. In particular, if $(A, I) \to (B, IB)$ is a flat map of prisms with (A, I) bounded, then (B, IB) is also bounded.

Lemma 3.11. Let $(C, IC) \xleftarrow{c} (A, I) \xrightarrow{b} (B, IB)$ be maps of bounded prisms where b is faithfully flat. Then the pushout of b along c exists in the category of prisms and it is a faithfully flat map of bounded prisms.

Proof. Take D to be the (p, I)-completion of $B \otimes_A^L C$. Then D is (p, I)-completely flat over A as b is flat. Then D is concentrated in degree zero, $D[I] = 0$, and D/ID has bounded p^∞-torsion, by Lemma 3.9. It follows that (D, ID) is a bounded prism by Lemma 3.6. It is clear that (D, ID) has the desired universal property. □

Remark 3.12. In particular, Lemma 3.11 shows that faithfully maps generate a Grothendieck topology on the category of bounded prisms, which we refer to as the flat topology.

Lemma 3.13. Let $(A, I) \to (B, IB)$ be a faithfully flat map of bounded prisms. Let $(B^\bullet, IB^\bullet)$ be the Čech nerve of $(A, I) \to (B, IB)$. Then we have $A \simeq \lim_\Delta B^\bullet$ as animated rings.

Proof. We have

$$\lim_{\Delta} B^\bullet = \lim_{[n]\in\Delta} \left(B^{\otimes_A (n+1)}\right)^\wedge_{(p,I)} = \lim_{[n]\in\Delta} \lim_m B^{\otimes_A (n+1)} /\!\!/ (p, I)^m = \lim_m A /\!\!/ (p, I)^m = A$$

where the third equality follows from faithfully flat descent of animated rings. □

Remark 3.14. This descent property shows that the flat topology on the category of bounded prisms, or the prismatic site, is subcanonical.

Lemma 3.15. The assignment $(A, I) \mapsto \widehat{D}_{\text{flat}}(A)$ carrying a bounded prism (A, I) to the category of (p, I)-complete (p, I)-completely flat A-modules is a sheaf on the category of bounded prisms with the flat topology.

Proof. This follows from (p, I)-completely faithfully flat descent. □

3.3 Perfect prisms

Lemma 3.16. The following two categories are equivalent.

(1) The category of perfectoid rings.

(2) The category of perfect prisms.

The functors are $R \mapsto (A_{\text{inf}}(R), \ker(\theta))$ and $(A, I) \mapsto A/I$ respectively.

Proof. Omitted. □

Lemma 3.17. Let (A, I) be a perfect prism corresponding to a perfectoid ring $R = A/I$. Let (B, J) be a prism. Then every map $A/I \to B/J$ of rings lifts uniquely to a map $(A, I) \to (B, J)$ of prisms.

Proof. Omitted. □

3.4 Prismatic envelopes

Lemma 3.18. Let (A, I) be a prism. The forgetful functor from the category of prisms over (A, I) to the category of δ-pairs over (A, I) admits a left adjoint.

Proof. Assume that $I = (d)$ is principal. Let $(A, (d)) \to (B, J)$ be a map of δ-pairs. Let $B_0 = A\{J/d\}^\wedge$ and let B_1 be the (p, d)-completion of the maximal d-torsion-free quotient of B_0. If $B_1[d] = 0$, then (B_1, dB_1) is a prism by Lemma 3.6, and (B_1, dB_1) has the the desired universal property. Otherwise, let B_2 be the (p, d)-completion of the maximal d-torsion-free quotient of B_1. This (transfinite) operation terminates because a countably filtered colimit of (p, d)-complete rings is (p, d)-complete. □

Lemma 3.19. Let (A, I) be a bounded prism. Let $(A, I) \to (B, J)$ be a map of δ-pairs where B is a (p, I)-complete (p, I)-completely flat δ-A-algebra. Assume that $J = (I, x_1, \dots, x_r)$ for a sequence $x_1, \dots, x_r \in B$ that is (p, I)-completely regular relative to A. Let $(A, I) \to (C, IC)$ be the prismatic envelope of $(A, I) \to (B, J)$.

(1) The ring C is (p, I)-completely flat over A. In particular, the prism (C, IC) is bounded.

(2) The construction $(B, J) \mapsto (C, IC)$ commutes with base-change along any map $(A, I) \to (A', IA')$ of bounded prisms.

(3) The construction $(B, J) \mapsto (C, IC)$ commutes with flat localization on B.

Proof. By Lemma 3.9 and Lemma 3.15, we may assume that $I = (d)$ is principal. Take

$$C = B\{x_1/d, \dots, x_r/d\}^\wedge,$$

formed in the category of animated δ-A-algebras. It remains to check that C is (p, d)-completely flat over A, and it follows immediately that C is discrete, together with other desired properties.

Consider the following diagram obtained by forming pushouts in the ∞-category of (p, d)-complete animated δ-rings

$$\begin{array}{ccccccc}
\mathbb{Z}_{(p)}\{z\}^\wedge & \xrightarrow{z \mapsto d} & A & \longrightarrow & B & \longrightarrow & B\{x_1/d, \dots, x_r/d\}^\wedge \\
\downarrow{\scriptstyle z \mapsto \phi(y)} & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}_{(p)}\{y\}^\wedge & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & B'\{x_1/\phi(y), \dots, x_r/\phi(y)\}^\wedge \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & A'\{\phi(y)/p\}^\wedge & \longrightarrow & B'' & \longrightarrow & B''\{x_1/p, \dots, x_r/p\}^\wedge.
\end{array}$$

Note that the bottom right term $B''\{x_1/p, \dots, x_r/p\}^\wedge$ is formed by Lemma 2.36 and the fact that $\phi(y) = d$ is distinguished. The map $\mathbb{Z}_{(p)}\{z\}^\wedge \to \mathbb{Z}_{(p)}\{y\}^\wedge$ is (p, z)-completely faithfully flat by Lemma 2.37 and Lemma 2.17. It then suffices to show that the map

$$A' \to B'\{x_1/\phi(y), \dots, x_r/\phi(y)\}^\wedge$$

is (p, z)-completely flat. By Lemma 2.39, the ring $A'\{\phi(y)/p\}^\wedge$ identifies with the p-completion of $D_{(y)}(A')$, and thus with the p-completion of the subring of $A'[1/p]$ generated by A' and $\{\gamma_n(y); n \geq 1\}$. It follows that the natural map $A' \to A' /\!\!/ (p, y)$ factors through $A'\{\phi(y)/p\}^\wedge$. By Lemma 2.38 and the fact that being (p, d)-completely regular is preserved under (p, d)-completed base-change, the map

$$A'\{\phi(y)/p\}^\wedge \to B''\{x_1/p, \dots, x_r/p\}^\wedge$$

at the bottom is (p, y)-completely flat. Then

$$B'\{x_1/\phi(y), \dots, x_r/\phi(y)\}^\wedge \otimes_{A'} A' /\!\!/ (p, y) \simeq B''\{x_1/p, \dots, x_r/p\}^\wedge \otimes_{A'\{\phi(y)/p\}^\wedge} A' /\!\!/ (p, y)$$

is discrete and flat over $A'/(p, y)$. Therefore $B'\{x_1/\phi(y), \dots, x_r/\phi(y)\}^\wedge$ is (p, y)-completely flat over A', as desired. □

Lemma 3.20. Let (A, I) be a bounded prism. Let R be a p-completely smooth A/I-algebra. Let $B_0 \to R$ be a surjection with B_0 being a (p, I)-completely smooth A-algebra. Let $B_0 \to B$ be a (p, I)-completely flat map with B being a (p, I)-complete δ-A-algebra. Let J_0 be the kernel of $B_0 \to R$. Let $J \subset B$ be the (p, I)-completion of $J_0 B$. Then the map $(A, I) \to (B, J)$ of δ-pairs satisfies the conditions of Lemma 3.19.

Proof. This follows from the standard properties of smoothness. ☐

4 The prismatic site

4.1 Basics

Let (A, I) be a bounded prism. Let R be an A/I-algebra.

Definition 4.1. We define a category $(R/A)_{\Delta}$ as follows.

(1) The objects of $(R/A)_{\Delta}$ are pairs (B, u) where (B, IB) is a bounded prism over (A, I), and $u : R \to B/IB$ is a map of A/I-algebras. Such an object is often denoted by $B \to B/IB \leftarrow R$.

(2) A morphism $(B, u) \to (C, v)$ in $(R/A)_{\Delta}$ is a map $f : B \to C$ of δ-A-algebras such that the diagram

$$\begin{array}{ccc} R & \xrightarrow{u} & B/IB \\ \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle f} \\ R & \xrightarrow{v} & C/IC \end{array}$$

commutes. Note that f is automatically a map of prisms over (A, I).

A map $(B, u) \to (C, v)$ in $(R/A)_{\Delta}$ is called a flat cover if the map $(B, IB) \to (C, IC)$ of prisms is faithfully flat, i.e. the ring map $B \to C$ is (p, I)-completely faithfully flat.

The prismatic site of R over (A, I) is the category $(R/A)_{\Delta}^{\mathrm{op}}$ equipped with the Grothendieck topology given by flat covers.

Remark 4.2. The category $(R/A)_{\Delta}$ is not a small category. However, there will not be genuine set-theoretical issues, as all the presheaves we consider are already sheaves.

Definition 4.3. We define the presheaf $\mathcal{O}_{\Delta}$ on $(R/A)_{\Delta}^{\mathrm{op}}$ valued in A-algebras by the assignment $(B, u) \mapsto B$. It is a sheaf by Lemma 3.13. It is called the structure sheaf of the prismatic site $(R/A)_{\Delta}^{\mathrm{op}}$. The prismatic complex of R over (A, I) is

$$\Delta_{R/A} = R\Gamma((R/A)_{\Delta}^{\mathrm{op}}, \mathcal{O}_{\Delta}) \in D(A).$$

We set $\overline{\Delta}_{R/A} = \Delta_{R/A} \otimes_A^L A/I$.

Remark 4.4. We mainly use the above definition if R is a smooth A/I algebra, in which case it coincides with the Kan extended version, see below.

Definition 4.5. Consider the functor $R \mapsto \Delta_{R/A}$ from the category of finite polynomial A/I-algebras to the category of commutative algebra objects in the ∞-category of (p, I)-complete objects in $D(A)$ equipped with a ϕ_A-semilinear endormophism. It can be extended to

$$\Delta_{\bullet/A} : \mathrm{AniAlg}_{A/I} \to \widehat{D}_{\phi_A}(A).$$

It is called derived prismatic cohomology in [BS22, Construction 7.6].

4.2 The Čech–Alexander complex

Let (A, I) be a bounded prism. Let R be a p-completely smooth A/I-algebra. Let $B_0 \to R$ be a surjection of A-algebras with B_0 being the (p, I)-completion of a polynomial A-algebra.

Construction 4.6. We shall construct a weakly initial object of $(R/A)_{\Delta}$.

Let B be the (p, I)-completion of the free δ-A-algebra on B_0. Let J_0 be the kernel of $B_0 \to R$. Let R' be the p-completion of B/J_0B. Let J be the kernel of $B \to R'$. Let (C, IC) be the prismatic envelope of (B, J) over (A, I). The situation is summarized in the following commutative diagram.

$$\begin{array}{ccccccc} A & \longrightarrow & B_0 & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A/I & \longrightarrow & R & \longrightarrow & R' & \longrightarrow & C/IC \end{array}$$

In particular, we obtain an object $(C, u : R \to C/IC)$ of the category $(R/A)_{\Delta}$.

Let (D, v) be another object of $(R/A)_{\Delta}$. As D is (p, I)-complete and B_0 is the (p, I)-completion of a polynomial A-algebra, by the universal property of (p, I)-completion, there exists an A-algebra map $B_0 \to D$ that lifts the A/I-algebra map $R \to D/ID$, see Lemma 2.13 and the following diagram

$$\begin{array}{ccccc} A & \longrightarrow & B_0 & \longrightarrow & D \\ \downarrow & & \downarrow & & \downarrow \\ A/I & \longrightarrow & R & \longrightarrow & D/ID. \end{array}$$

The map $B_0 \to D$ extends to an A-algebra map $B \to D$, as D is a δ-A-algebra and B is the free δ-A-algebra on B_0. Since the square on the left

$$\begin{array}{ccccc} B_0 & \longrightarrow & B & \longrightarrow & D \\ \downarrow & & \downarrow & & \downarrow \\ R & \longrightarrow & R' & \dashrightarrow & D/ID. \end{array}$$

is a pushout, there exists a unique dashed map $R' \to D/ID$ fitting into the diagram. In particular, the map $B \to D$ is a map $(B, J) \to (D, ID)$ of δ-pairs, and thus by the universal property of prismatic envelopes, it induces a unique map $(C, IC) \to (D, ID)$ of prisms over (A, I). It follows that the map induces a map $(C, u) \to (D, v)$ in $(R/A)_{\Delta}$ as $R = B_0/J_0$.

Therefore the object $(C, u) \in (R/A)_{\Delta}$ is weakly initial.

Construction 4.7. We shall construct a cosimplicial A-algebra computing the complex $\Delta_{R/A}$.

Let $B_0^\bullet$ be the (p, I)-completion of the Čech nerve of $A \to B_0$. Apply Construction 4.6 termwise on $B_0^\bullet \to R$, and we obtain a cosimplicial object $(C^\bullet \to C^\bullet/IC^\bullet \leftarrow R)$ of $(R/A)_{\Delta}$. The functoriality of Construction 4.6 shows that the resulting cosimplicial object identifies with the Čech nerve of $(C^0 \to C^0/IC^0 \leftarrow R)$. By Lemma 2.21, we conclude that $C^\bullet$ computes $\Delta_{R/A}$. It follows that $C^\bullet/IC^\bullet$ computes $\overline{\Delta}_{R/A}$.

4.3 Permanence properties

Let (A, I) be a bounded prism.

Lemma 4.8. Let $R \to S$ be a p-completely étale map of p-completely smooth A/I-algebras.

(1) The natural functor $(S/A)_{\Delta} \to (R/A)_{\Delta}$ admits a left adjoint F.

(2) Then the natural map

$$\overline{\Delta}_{R/A} \otimes_R^L S \to \overline{\Delta}_{S/A}$$

is an isomorphism.

Proof. Proof of (1). Let $B \to B/IB \leftarrow R$ be an object of $(R/A)_{\Delta}$. By Elkik's algebraization, we choose an étale R-algebra S_0 such that S is the p-completion of S_0. By [Sta, Tag 04D1], we can choose an étale B-algebra C_1 such that $C_1/IC_1 \simeq B/IB \otimes_R S_0$. Let C be the (p, I)-completion of C_1. Then C is (p, I)-complete (p, I)-completely étale over B, and thus it admits a unique δ-structure compatible with B by Lemma 2.40. So we obtain an object

$$C \to C/IC \simeq B/IB \,\widehat{\otimes}^L_R\, S \leftarrow S$$

of $(S/A)_{\Delta}$ satisfying the desired universal property.

Proof of (2). Choose a surjection $B_0 \to R$ with B_0 being (p, I)-completion of a polynomial A-algebra, and we obtain a cosimplicial object $(C^\bullet \to C^\bullet/IC^\bullet \leftarrow R)$ by Construction 4.7. Let

$$(D^\bullet \to D^\bullet/ID^\bullet \leftarrow S) = F(C^\bullet \to C^\bullet/IC^\bullet \leftarrow R).$$

Then $(D^\bullet \to D^\bullet/ID^\bullet \leftarrow S)t$ is the Čech nerve of the weakly initial object $(D^0 \to D^0/ID^0 \leftarrow S)$, and thus $D^\bullet/ID^\bullet$ computes $\overline{\Delta}_{S/A}$. The construction of F in (1) shows that $C^\bullet/IC^\bullet \otimes^L_R S \simeq D^\bullet/ID^\bullet$. We conclude by Lemma 2.18. ☐

Lemma 4.9. Let $(A, I) \to (A', IA')$ be a map of bounded prisms such that the ring map $A \to A'$ is of finite (p, I)-complete tor-amplitude. Let $R' = R \,\widehat{\otimes}_A\, A'$ be the (p, I)-completed base-change. Then we have a natural isomorphism

$$\Delta_{R/A} \,\widehat{\otimes}_A\, A' \simeq \Delta_{R'/A'}$$

and the similar holds for $\overline{\Delta}_{R/A}$.

Proof. Use Construction 4.7 to obtain a cosimplicial A-algebra $C^\bullet$ computing $\Delta_{R/A}$, apply $- \widehat{\otimes}^L_A A'$ termwise to $C^\bullet$, and finish using Lemma 2.18. ☐

5 Comparision theorems

5.1 The crystalline comparison

In this section we prove the crystalline comparison theorem, first for the pd-thickening $A \to A/p$, and then the general situation. Although both proof involve some choices, the resulting comparison isomorphism is canonical, and we omit relevant discussions (see Remark 5.6).

Definition 5.1. A prism (A, I) is called cyrstalline if $I = (p)$.

Remark 5.2. A δ-pair $(A, (p))$ is crystalline if and only if A is p-torsion-free and p-complete.

Lemma 5.3. Let $(A, (p))$ be a cyrstalline prism. Let R be a smooth A/p-algebra. Let $R^{(1)}$ be the pullback of R along the Frobenius $A/p \to A/p$. Then we have a natural isomorphism

$$\Delta_{R^{(1)}/A} \simeq R\Gamma_{\mathrm{crys}}(R/A)$$

of $\mathbb{E}_\infty$-A-algebras, i.e. it is an isomorphism of commutative algebras in $D(A)$.

Proof. Choose a surjection $B_0 \to R$ of A-algebras where B_0 is the p-completion of a polynomial A-algebra (for example the polynomial algebra on R). We obtain cosimplicial A-algebras $D^\bullet$ and $C^\bullet$ computing the crystalline cohomology $R\Gamma_{\mathrm{crys}}(R/A)$ and the prismatic cohomology $\Delta_{R/A}$, respectively, using Construction 2.27 and Consruction 4.7. By a close inspection on the construction of the Čech–Alexander complex, it holds that $\phi_A^* C^\bullet$, i.e. the termwise pullback along the associated Frobenius, computes $\Delta_{R^{(1)}/A}$. Recall that $D^\bullet$, by definition, is the (termwise) p-completion of the pd-envelope $D_{J^\bullet}(B^\bullet)$. By Lemma 2.39, we have a natural isomorphism

$$D^\bullet \simeq B^\bullet\{\phi(J^\bullet)/p\}^\wedge.$$

By Lemma 3.19, we know that $C^\bullet$, as the prismatic envelope of $(B^\bullet, J^\bullet)$, identifies with $B^\bullet\{J^\bullet/p\}^\wedge$. Hence we obtain a natural comparison map

$$\eta : \phi_A^* C^\bullet \simeq \phi_A^*(B^\bullet\{J^\bullet/p\}^\wedge) \to B^\bullet\{\phi(J^\bullet)/p\}^\wedge \simeq D^\bullet$$

of cosimplicial A-algebras. Moreover, the cosimplicial A-algebra $D^\bullet$ can be identified with $\phi_A^* C^\bullet \otimes_{B^\bullet,(1)} B^\bullet \simeq \phi_{B^\bullet}^* C^\bullet$. Applying Lemma 2.25, we see that η induces a quasi-isomorphism after modulo p. It follows that η is a quasi-isomorphism by Lemma 2.14. □

Remark 5.4. Let A be a ring. Let I be a pd-ideal of A such that $p \in I$. For $x \in I$, we have $x^p = p! \cdot \gamma_p(a)$, cf. [Sta, Tag 07GM]. It follows that the absolute Frobenius $A/p \to A/p$ induces a ring map $\psi : A/I \to A/p$.

Lemma 5.5. Let $(A, (p))$ be a crystalline prism. Let I be a pd-ideal of A with $p \in I$. Let R be a smooth A/I-algebra. Let $R^{(1)} = R \otimes_{A/I} A/p$ where A/p is regarded as an A/I-algebra via the map ψ. Then there is a natural isomorphism $\Delta_{R^{(1)}/A} \simeq R\Gamma_{\mathrm{crys}}(R/A)$ of $\mathbb{E}_\infty$-A-algebras.

Proof. The natural map $A/p \to A/I$ is surjective, and thus we can choose a smooth A/p-algebra $\widetilde{R}$ lifting R by [Sta, Tag 07M8]. By Lemma 5.3, we have the isomorphism

$$\Delta_{\widetilde{R}^{(1)}/A} \simeq R\Gamma_{\mathrm{crys}}(\widetilde{R}/A).$$

Note that $\widetilde{R}^{(1)} \simeq R^{(1)}$ as the Frobenius $A/p \to A/p$ is the composition $A/p \to A/I \to A/p$. By [Ber06, Théorème 3.5.1], we have a canonical isomorphism $R\Gamma_{\mathrm{crys}}(\widetilde{R}/A) \simeq R\Gamma_{\mathrm{crys}}(R/A)$. Therefore we obtain a functorial isomorphism $\Delta_{R^{(1)}/A} \simeq R\Gamma_{\mathrm{crys}}(R/A)$. □

Remark 5.6. We now explain briefly why the comparision map is canonical.

(1) In Lemma 5.3, a map $B_0 \to R$ is chosen. The collection of such maps $B_0 \to R$ forms a sifted category, and it is weakly contractible when viewed as an ∞-category, cf. [Lura, Tag 02QL]. Hence the map to the mapping anima $\mathrm{Map}_{\mathrm{Alg}(D(A))}(\Delta_{R^{(1)}/A}, R\Gamma_{\mathrm{crys}}(R/A))$ is null-homotopic, cf. [Lura, Tag 050U].

(2) In Lemma 5.5, we choose a lift $\widetilde{R}$. The resulting comparison map is actually compatible with a Čech–Alexander style construction in (1). We omit the details here and refer the readers to [BS22].

Remark 5.7. The canonical comparison map $\eta : \Delta_{R^{(1)}/A} \simeq R\Gamma_{\mathrm{crys}}(R/A)$ is compatible with the Frobenius, see the proof of [BS22, Theorem 5.2].

5.2 The Hodge–Tate comparison

Let (A, I) be a bounded prism. We prove the Hodge–Tate comparison in three steps.

(1) The first step is to prove the characteristic p case, which is deduced from the crystalline comparison.

(2) The second step is to prove the comparison for affine spaces by reducing to (1). Then we can construct the Hodge–Tate comparison map for smooth A/I-algebras.

(3) The last step is to prove that the Hodge–Tate comparison map is an isomorphism by reducing to (2).

Lemma 5.8. Let R be an $\mathbb{F}_p$-algebra. Let S be a finite polynomial R-algebra. Let $\phi : S^{(1)} \to S$ be the relative Frobenius. Then there is a quasi-isomorphism of $S^{(1)}$-dga

$$(\Omega^\bullet_{S^{(1)}/R}, 0) \to (\Omega^\bullet_{S/R}, d_{\mathrm{dR}})$$

extending ϕ (in degree zero).

Proof. This is clear using a basis. Alternatively, this is a special case of [Kat70, Theorem 7.2]. □

Lemma 5.9. Let $(A, (p))$ be a crystalline prism. Let S be a smooth A/p-algebra. Then we have an isomorphism

$$H^i(\overline{\Delta}_{S/A})\{i\} \simeq \Omega^i_{S/(A/p)}.$$

Proof. By the standard properties of smoothness (say [Sta, Tag 054L]) and Lemma 4.8, we may assume that S is a finite polynomial algebra over A/p. By Lemma 4.9, we reduce to the case $A = \mathbb{Z}_p$. Then $S = R^{(1)}$ for $R = \mathbb{F}_p[X_1, \ldots, X_n]$. We have $\Delta_{S/A} \simeq R\Gamma_{\mathrm{crys}}(R/A)$ by the crystalline comparison, and thus

$$\overline{\Delta}_{S/A} \simeq R\Gamma_{\mathrm{crys}}(R/A) \otimes^L_A A/p \simeq \Omega_{R/(A/p)}$$

by the comparison between crystalline cohomology and de Rham cohomology. Therefore the claim follows from Lemma 5.8. ☐

Construction 5.10. Let $(A, (d))$ be then universal oriented prism, i.e. A is the (p, d)-completion of $A_0 = \mathbb{Z}_{(p)}\{d, \delta(d)^{-1}\}$ and $I = (d)$. Note that A/d is p-torsion-free and the Frobenius on A/p is flat. Let $B = A\{\phi(d)/p\}^\wedge$ be the p-completed simplicial δ-A-algebra obtained by freely adjoining $\phi(d)/p$ to A, cf. Lemma 2.39. It follows that the A-algebra B is discrete, $(p, \phi(d))$-complete, (p, d)-complete, p-torsion-free, and identifies withthe p-completed pd-envelope of $(d) \subset A$. Let $\alpha : A \to B$ be the composition of the structure map $A \to B$ and the associated Frobenius $\phi : A \to A$.

Remark 5.11. The map α/p can be factored as

$$A/p \to A/(p, d) \xrightarrow{\phi} A/(p, d^p) \to B/p \simeq D_{(d)}(A/p).$$

Note that the first map is of finite d-complete tor-amplitude, and that the seond and the thirde map are both faithfully flat. It follows that the completed base-change functor

$$\widehat{\alpha}^* : \widehat{D}(A) \to \widehat{D}(B)$$

on the (p, d)-complete objects of the derived categories has the following properties.

(1) The functor $\widehat{\alpha}^*$ is conservative.

(2) The functor $\widehat{\alpha}^*$ has finite (p, d)-complete tor-amplitude.

(3) The functor $\widehat{\alpha}^*$ preserves prismatic cohomology, i.e. for every p-completely smooth A/I-algebra R with p-completed base-change R_B to B/IB, we have an isomorphism $\widehat{\alpha}^* \Delta_{R/A} \simeq \Delta_{R_B/B}$.

Construction 5.12. Let R be a p-completely smooth A/I-algebra. We have a fibre sequence

$$\Delta_{R/A} \otimes^L_A I^{i+1}/I^{i+2} \to \Delta_{R/A} \otimes^L_A I^i/I^{i+2} \to \Delta_{R/A} \otimes^L_A I^i/I^{i+1}$$

induced from $I^{i+1}/I^{i+2} \to I^i/I^{i+2} \to I^i/I^{i+1}$. The long exact sequence gives maps

$$\beta : H^i(\overline{\Delta}_{R/A})\{i\} \to H^{i+1}(\overline{\Delta}_{R/A})\{i+1\}$$

for every $i \geq 0$. Thus we obtain a differential graded algebra $H^\bullet(\overline{\Delta}_{R/A})\{\bullet\}$ over A/I. We have the natural map

$$\eta^0 : \Omega^0_{R/(A/I)} = R \to H^0(\overline{\Delta}_{R/A})\{0\}.$$

By the universal property of Kähler differentials, we obtain a map of R-modules

$$\eta^1 : \Omega^1_{R/(A/I)} \to H^1(\overline{\Delta}_{R/A})\{1\}$$

corresponding to the derivation $\beta \circ \eta^0$.

Lemma 5.13. Let $R = (A/I)[X]^\wedge$.

(1) Both $\eta^0 : R \to H^0(\overline{\Delta}_{R/A})$ and $\eta^1 : \Omega^1_{R/(A/I)} \to H^1(\overline{\Delta}_{R/A})\{1\}$ are isomorphisms.

(2) $H^i(\overline{\Delta}_{R/A}) = 0$ for $i > 1$.

Proof. We start with several reductions. First, we can reduce to the case that (A, I) is orientable by Lemma 3.9. Choose an orientation $I = (d)$. Let $\alpha : A_0 \to B_0$ be the map defined in Construction 5.10 where A_0 is the universal oriented prism. We have a unique map $A_0 \to A$ by the universal property. Let $B = A \widehat{\otimes}^L_{A_0} B_0$. By Remark 5.11, the functor $\widehat{\beta}^* : \widehat{D}(B) \to \widehat{D}(A)$ is conservative. The complex $\widehat{\beta}^* \overline{\Delta}_{R/A}$ admits an explicit description (by chosing a suitable cosimplicial object computing $\overline{\Delta}_{R/A}$, e.g. [BS22, Remark 4.19]) as the completed pullback of $\overline{\Delta}_{\mathbb{F}_p[X]/\mathbb{Z}_p}$ along the natural map $\mathbb{Z}_p \to B$. Hence it suffices to prove the claim for $\overline{\Delta}_{\mathbb{F}_p[X]/\mathbb{Z}_p}$. In this special case, the base prism is $(A, (p))$. Then we are done by Lemma 5.9. □

Remark 5.14. The above proof obviously generalizes to $R = (A/I)[X_1, \ldots, X_n]^\wedge$, in the sense that $H^d(\overline{\Delta}_{R/A}) = 0$ for $d > n$ and for $d \leq n$ the comparison map η^d is an isomorphism, once the correpsonding comparison maps are defined (which is not the case at this point).

Lemma 5.15. Let R be a p-completely smooth A/I-algebra. Let $f \in R$. Then $\beta(f) \in H^1(\overline{\Delta}_{R/A})\{1\}$ squares to zero in $H^2(\overline{\Delta}_{R/A})\{2\}$.

Proof. Note that the element f corresponds to a map $(A/I)[X]^\wedge \to R$, and then the result follows from Lemma 5.13 by functoriality. □

Remark 5.16. Let $B \to C$ be a ring map. Recall that the de Rham complex $\Omega^\bullet_{C/B}$, viewed as a strictly commutative differential graded algebra over B, has the following universal property. Let $E^\bullet$ be a commutative B-dga with $E^i = 0$ for $i < 0$. Let $\eta : C \to E^0$ be a B-algebra map such that for every $x \in C$, the element $y = d(\eta(x)) \in E^1$ satisfies $y \cdot y = 0$ (note that this is automatic if $E^\bullet$ is strictly graded commutative, see [Sta, Tag 061W]). Then the map $C \to E^0$ extends uniquely to a map $\Omega^\bullet_{C/B} \to E^\bullet$ of B-dgas.

Construction 5.17. Combining Lemma 5.15 and Remark 5.16, we obtain the Hodge–Tate comparison

$$\eta^\bullet : \Omega^\bullet_{R/(A/I)} \to H^\bullet(\overline{\Delta}_{R/A})\{\bullet\}$$

of commutative differential graded R-algebras, for a p-completely smooth A/I-algebra R.

Lemma 5.18. Let R be a p-completely smooth A/I-algebra. Then the Hodge–Tate comparison map

$$\eta^\bullet : \Omega^\bullet_{R/(A/I)} \to H^\bullet(\overline{\Delta}_{R/A})\{\bullet\}$$

is an isomorphism of A/I-dgas.

Proof. This is clear, see Remark 5.14. □

5.3 The étale comparison

Let $(A, (d))$ be a perfect prism. Let S be a p-complete A/d-algebra. Write F and G for the functors on p-complete A/d-algebras

$$F(S) = R\Gamma(\mathrm{Spec}(S[1/p]), \mathbb{Z}/p^n), \quad G(S) = (\Delta_{S/A}[1/d]/p^n)^{\phi=1}.$$

In this section we sketch the construction of the étale comparison map $F \to G$, i.e. a functorial isomorphism

$$R\Gamma(\mathrm{Spec}(S[1/p]), \mathbb{Z}/p^n) \to (\Delta_{S/A}[1/d]/p^n)^{\phi=1}.$$

Remark 5.19. By Lemma 2.46, the functor F is a sheaf for the arc-p topology.

Lemma 5.20. The functor G is a sheaf for the arc-p topology.

Proof. We first introduce the perfection of $\Delta_{S/A}$, defined as

$$\Delta_{S/A,\mathrm{perf}} = \mathrm{colim}\Big[\Delta_{S/A} \xrightarrow{\phi_S} \Delta_{S/A} \xrightarrow{\phi_S} \Delta_{S/A} \to \cdots\Big]^\wedge.$$

It is a (p, I)-complete $\mathbb{E}_\infty$-A-algebras. The map

$$(\Delta_{S/A}[1/d]/p)^{\phi=1} \to (\Delta_{S/A}[1/d]/p)^{\phi=1}$$

induced by ϕ_S is an isomorphism, and thus by the fact that taking fixed points of the Frobenius commutes with completed colimits ([BS22, Lemma 9.2]), we have an isomorphism

$$(\Delta_{S/A}[1/d]/p^n)^{\phi=1} \to (\Delta_{S/A,\mathrm{perf}}[1/d]/p^n)^{\phi=1}$$

for every $n \geq 1$ functorial in S. The functor $S \mapsto (\Delta_{S/A,\mathrm{perf}}[1/d]/p^n)^{\phi=1}$ is a sheaf for the arc topology by [BS22, Corollary 8.11], and we can proceed as [BM21, Corollary 6.17] to show that G is in fact a sheaf for the arc-p topology. □

Construction 5.21. We shall construct a map $F \to G$. As the arc-p topology is finer than the Zariski topology, we obtain a map of functors

$$H^0(F(-)) = H^0(\mathrm{Spec}((-)[1/p]), \mathbb{Z}/p^n) \to H^0_{\text{arc-}p}(-, \mathbb{Z}/p^n).$$

It induces a map of sheaves

$$F \to R\Gamma_{\text{arc-}p}(-, \mathbb{Z}/p^n)$$

because the sheaf F (valued in the derived category) is actually equal to the sheafification of the presheaf $H^0(F(-))$ (note that sheafification valued in the derived category computes sheaf cohomology). On the other hand, the obvious map $\mathbb{Z}/p^n \to G$ of presheaves induces a map

$$R\Gamma_{\text{arc-}p}(-, \mathbb{Z}/p^n) \to G.$$

Combining the above two maps, and we obtain the étale comparison map $F \to G$.

Lemma 5.22. The canonical map $F \to G$ in Construction 5.21 is an isomorphism. In particular, we have a canonical isomorphism

$$R\Gamma(\mathrm{Spec}(S[1/p]), \mathbb{Z}/p^n) \simeq (\Delta_{S/A}[1/d]/p^n)^{\phi=1}$$

for each $n \geq 1$.

Proof. By arc-p descent, we reduce to the case $S = \prod R_i$ where every R_i is an absolutely integrally closed valuation ring of rank ≤ 1, cf. [BS22, Remark 8.9] and [BM21, Proposition 3.30]. In this case, $F(S)$ is equal to the product of copies of $\mathbb{Z}/p^n$, and $G(S)$ can be identified with $W(S^\flat)$ (see [BS22, Example 8.3]). The desired isomorphism then follows from the Artin–Schreier–Witt exact sequence. □

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