Étale Cohomology

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1 Introduction

Let M be a complex manifold. One question is how to study the geometry of M. One possible approach is to put a metric on M. Another possibility is to "linearize" M, i.e. using cohomology theories. Recall that we have the following cohomology theories.

1. *Betti cohomology* $H_B^*(M, \mathbb{Z})$ (which could be defined using singular cohomology or sheaf cohomology). This is a topological invariant.

2. *de Rham cohomology* $H_{\mathrm{dR}}^*(M)$.

There are two important theorems in complex geometry.

Lemma 1.1 (de Rham isomorphism)**.** $H_B^*(M, \mathbb{Z}) \otimes \mathbb{C} \simeq H_{\mathrm{dR}}^*(M)$.

Proof. Hint: use the holomorphic Poincaré lemma, i.e. the constant sheaf $\underline{\mathbb{C}}_M$ is quasi-isomorphic to the de Rham complex $\mathcal{O}_M \to \Omega_M^1 \to \cdots$. □

Note that de Rham cohomology is equipped with complex conjugation and a filration (coming from the hypercohomology).

Lemma 1.2 (Hodge decomposition)**.** If M is compact Kähler (for example M is projective), then we have a functorial decomposition

$$H_{\mathrm{dR}}^n(M) \simeq \bigoplus_{p+q=n} H^{p,q}(M)$$

where $H^{p,q}(M)$ is the (p,q)-Dolbeault cohomology, with $\overline{H^{p,q}} = H^{q,p}$.

We ask the following question. Are there algebraic analogues to these cohomology theories? For de Rham cohomology, we have algebraic de Rham cohomology. Let X bea smooth algebriac variety over a field k. The algebraic de Rham cohomology of X is

$$H_{\mathrm{dR}}^*(X/k) = \mathbb{H}^*(X, \mathcal{O}_X \to \Omega_X^1 \to \cdots).$$

It is a correct theory in the sense of the following result.

Lemma 1.3 (Grothendieck)**.** If $k = \mathbb{C}$, then we have a canonical isomorphism

$$H_{\mathrm{dR}}^*(X/\mathbb{C}) \to H_{\mathrm{dR}}^*(X(\mathbb{C}))$$

where $X(\mathbb{C})$ is regarded as a complex manifold.

Proof. Hint: use GAGA for the projective case, and the smooth case follows by standard techniques. □

However, algebraic de Rham cohomology has different behaviours in positive characteristics. For example, consider the affine line $X = \mathbb{A}^1_{\mathbb{F}_p} = \operatorname{Spec} \mathbb{F}_p[x]$. In this case, we have

$$H^*_{\mathrm{dR}}(X/\mathbb{F}_p) = H^*(\mathbb{F}_p[x] \to \mathbb{F}_p[x]dx)$$

where the map $\mathbb{F}_p[x] \to \mathbb{F}_p[x]dx$ is $f(x) \mapsto f'(x)dx$ (note that this is the map of global sections of $\mathcal{O}_X \to \Omega^1_X$). It follows that $H^0_{\mathrm{dR}}(\mathbb{A}^1/\mathbb{F}_p) = \mathbb{F}_p[x^p]$ which is not finite dimensional over $\mathbb{F}_p$.

How about an algebraic analogue of Betti cohomology? Another feature of the Betti cohomology is its relation with the fundamental group

$$H^1_B(M, \mathbb{Z}) \simeq \operatorname{Hom}(\pi_1(M), \mathbb{Z}).$$

An answer is the étale cohomology $H^*_{\text{ét}}(X, \mathcal{F})$ where $\mathcal{F}$ is some coefficient (for example $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$, but not $\mathbb{Z}$), and there is the étale fundamental group $\pi_1^{\text{ét}}(X)$.

Lemma 1.4 (Artin)**.** If $k = \mathbb{C}$, there is a natural isomorphism

$$H^n_B(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \simeq H^n_{\text{ét}}(X, \mathbb{Z}_\ell).$$

Moreover, we have

$$\pi_1^{\text{ét}}(X) \simeq \pi_1^{\text{top}}(X(\mathbb{C}))^\wedge$$

where we perform a profinite completion of the (topological) fundamental group.

Example 1.5. For the affine line, we have

$$H^n_{\text{ét}}(\mathbb{A}^1, \mathbb{Z}_\ell) = H^n_B(\mathbb{C}, \mathbb{Z}_\ell) = \begin{cases} \mathbb{Z}_\ell & n = 0 \\ 0 & n \neq 0, \end{cases}$$

and

$$\pi_1^{\text{ét}}(\mathbb{A}^1) = \pi_1^{\text{top}}(\mathbb{C})^\wedge = 0.$$

Another example is

$$\pi_1^{\text{ét}}(\mathbb{G}_m, \overline{1}) = \pi_1^{\text{top}}(\mathbb{C}^\times, 1)^\wedge = \widehat{\mathbb{Z}}(1).$$

Recall that the fundamental group $\pi_1(M)$ can be alternatively described as the automorphism group of the universal covering $\widetilde{M} \to M$. This gives a Galois theory for covering spaces. The completed version $\pi_1(M)^\wedge$ then corresponds to the Galois theory for finite covering spaces (which is more desired in the algebraic case, as it's really hard to write, say, the exponential map as a polynomial). So the first step to algebraize the theory is to define an algebraic notion of "local isomorphisms" of algebraic varieties. This is called *étale morphisms.*

Étale cohomology is a good cohomology theory for algebraic varieties even in characteristic p. Let X_0 be a smooth algebraic variety over $k = \mathbb{F}_q$ of characteristic p. Let $F_{X_0} : X_0 \to X_0$ be the (q-th) absolute Frobenius. Let $X = X_0 \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$. It is equipped with an $\overline{\mathbb{F}}_q$-scheme endomorphism $F = F_{X_0} \otimes \mathrm{id}_{\overline{\mathbb{F}}_q}$. For $\ell \neq p$, we obtain an endomorphism F^* of the finite dimensional $\mathbb{Q}_\ell$-vector space $H^n_{\text{ét}}(X, \mathbb{Q}_\ell)$. Note that $(X(\overline{\mathbb{F}}_p))^{F^n = \mathrm{id}} = X_0(\mathbb{F}_{q^n})$ for $n \geq 1$.

Lemma 1.6 (Grothendieck's Lefschetz fixed point theorem)**.** If X is proper over $\mathbb{F}_q$, then

$$\# X_0(\mathbb{F}_{q^n}) = \# X(\overline{\mathbb{F}}_q)^{F^n = \mathrm{id}} = \sum_{m=0}^{2 \dim X} (-1)^m \operatorname{tr}(F^{*,n}; H^m_{\text{ét}}(X, \mathbb{Q}_\ell)).$$

The generating series is

$$\sum_{n=1}^{\infty} \# X_0(\mathbb{F}_{q^n}) \frac{t^n}{n} = \sum_{m=0}^{2 \dim X_0} (-1)^m \sum_{n=1}^{\infty} \operatorname{tr}(F^{*,n}; H^m) \frac{t^n}{n}.$$

Apply the exponential map, and we obtain

$$\prod_{x\in|X_0|}\frac{1}{1-t^{\deg x}}=\prod_{m=0}^{2\dim X_0}\det\bigl(1-tF^*;H^m_{\text{ét}}(X,\mathbb{Q}_\ell)\bigr)^{(-1)^{m+1}}$$

where $|X_0|$ is the set of closed points of X_0. Hint: when computing the left hand side, we can show that

$$\#X_0(\mathbb{F}_{q^n})=\sum_{d|n}\sum_{\substack{x\in|X_0|\\ \deg x=d}}d.$$

We write $Z(X_0/\mathbb{F}_q,t)$ for the left hand side, and then we obtain the zeta function

$$Z(X_0/\mathbb{F}_q,q^{-s})=\prod_{x\in|X_0|}\frac{1}{1-(\#k(x))^{-s}}.$$

Lemma 1.7 (Deligne; Weil conjecture)**.** If X_0 is smooth and proper, then the polynomial

$$P_m(t)=\det(1-tF^*;H^m_{\text{ét}}(X,\mathbb{Q}_\ell))\in\mathbb{Q}_\ell[t]$$

lands in $1+t\mathbb{Z}[t]$ and is independent of ℓ. Moreover, if we write $P_m(t)=\prod(1-\alpha_{m,i}t)$ in $\mathbb{C}[t]$, then $|\alpha_{m,i}|=q^{m/2}$.

Here is another feature of étale cohomology. Let X be an algebraic variety over a field k. Let ℓ be different from the characteristic of k. The étale cohomology $H^n_{\text{ét}}(X_{\overline{k}},\mathbb{Q}_\ell)$ is equipped with a (continuous) action by the absolute Galois group Gal_k. This gives a general construction of continuous ℓ-adic Galois representations of k. This is important in Langlands program.

How about an algebraic analogue of the de Rham isomorphism? This is a little bit problematic as the objects are $\mathbb{Q}_\ell$-vector spaces and k-vector spaces respectively. However we have the following result from p-adic Hodge theory.

Lemma 1.8 (Tsuji, Faltings, Nizioł, ...)**.** Let k be a finite extension of $\mathbb{Q}_p$. Let X be a smooth proper algebraic variety over k. There is a canonical isomorphism

$$H^n_{\text{ét}}(X_{\overline{k}},\mathbb{Q}_p)\otimes_{\mathbb{Q}_p}B_{\text{dR}}\simeq H^n_{\text{dR}}(X/k)\otimes_k B_{\text{dR}}$$

where B_{dR} is Fontaine's de Rham period ring.

2 Étale Morphisms

Let $f:X\to Y$ be a morphism of schemes locally of finite type. Recall that $\Omega_{X/Y}$ is the sheaf of Kähler differentials, which is a finitely generated quasi-coherent $\mathcal{O}_X$-module.

Example 2.1. Locally, we have $X=\operatorname{Spec}B\to Y=\operatorname{Spec}A$, and $B=A[x_1,\dots,x_n]/I$. Then $\Omega_{X/Y}=\Omega^{\sim}_{B/A}$ and

$$\Omega_{B/A}=\frac{\oplus_{i=1}^n B\,dx_i}{(df\mid f\in I)}.$$

If $B=A[x]/(f(x))$, then $\Omega_{B/A}=A[x]/(f(x),f'(x))$. If $A=K$ is a field, and $B=L=K(\alpha)$ is a finite simple extension, then $\Omega_{B/A}=0$ if and only if $K(\alpha)/K$ is separable.

Lemma 2.2. Let $x\in X$ with image $y=f(x)\in Y$. The following are equivalent.

1. $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is a finite separable extension of $k(y)$. In particular, $\mathfrak{m}_x=\mathfrak{m}_y\mathcal{O}_{X,x}$.
2. We have $(\Omega_{X/Y})_x=0$.
3. The diagonal $\Delta_{X/Y}:X\to X\times_Y X$ is an open immersion on an open neighbourhood of x.

Proof. Proof of (2) ⇔ (3). By Nakayama's lemma, (2) is equivalent to $(\Omega_{X/Y})_x \otimes k(x) = 0$. Locally write $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, then $\Omega_{B/A} = J/J^2$ where J is the kernel of $B \otimes_A B \to B$. So (2) is equivalent to $J_w/J_w^2 = 0$ where $w = \Delta_{X/Y}(x)$, and to $J_w = 0$ by Nakayama's lemma again, and finally equivalent to that locally near x, $\Delta_{X/Y}$ is an isomorphism to its image.

Assume (1). Then $(\Omega_{X/Y})_x \otimes k(x) = \Omega_{(\mathcal{O}_x/\mathfrak{m}_y\mathcal{O}_x)/k(y)} = 0$.

Assume (2) and (3). We may assume $Y = \operatorname{Spec} k$ with $k = k(y)$, and $X = \operatorname{Spec} B$, and that $\Delta_{X/Y}$ is an open immersion. Then B is a finite k-algebra (hint: Noether normalization and (3)). Then $B = \prod B_i$ is a product of local finite k-algebra. It suffices to show that each B_i is a finite separable extension of k, or equivalently, that $B \otimes_k \overline{k}$ is isomorphic to a product of copies of $\overline{k}$. We may assume that $k = \overline{k}$. Hence for $\mathfrak{n} \in \operatorname{Spec} B$, the map $B_\mathfrak{n} \otimes_k B_\mathfrak{n} \to k$ is an isomorphism, and we conclude $B_\mathfrak{n} = k$ by dimension reason. □

Definition 2.3. Let $f : X \to Y$ be locally of finite presentation. Let $x \in X$.

1. We say that f is *unramified* at x (in the sense of EGA IV) if the equivalent conditions in the previous lemma holds for x.

2. We say that f is *étale* at x if f is unramified at x and flat at x.

3. We say that f is unramified or étale if it is so at every point of X.

Lemma 2.4. First properties.

1. Local immersions which are of finite presentation are unramified. Open immersions (which are automatically of finite presentation) are étale.

2. Unramified (or étale morphisms) are stable under compositions and base-changes.

3. Let $f : X \to Y$ and $g : Y \to Z$ be locally of finite presentation. If $g \circ f$ is unramified, then f is unramified. If $g \circ f$ is étale, and g is unramified, then f is étale.

Proof. Exercise. □

Example 2.5. Let $L/K/\mathbb{Q}$ be finite field extensions. This gives a flat morphism $f : \operatorname{Spec} \mathcal{O}_L \to \operatorname{Spec} \mathcal{O}_K$. Let $\mathfrak{q} \in \operatorname{Spec} \mathcal{O}_L$ and $\mathfrak{p} = f(\mathfrak{q}) = \mathfrak{q} \cap \mathcal{O}_K$. Then f is unramified at $\mathfrak{q}$ if and only if $\mathcal{O}_{L,\mathfrak{q}}/\mathfrak{p}\mathcal{O}_{L,\mathfrak{q}}$ is a finite separable extension of $\mathcal{O}_K/\mathfrak{p}$, if and only if $\mathfrak{p}\mathcal{O}_{L,\mathfrak{q}} = \mathfrak{q}\mathcal{O}_{L,\mathfrak{q}}$, i.e. $e(\mathfrak{q}/\mathfrak{p}) = 1$, i.e. $\mathfrak{q}$ is unramified over $\mathfrak{p}$.

Example 2.6. Let k be a field. Let $f : \mathbb{A}^1_k \to \mathbb{A}^1_k$ be the map $A = k[y] \to B = k[x]$ defined by $y \mapsto x^n$. Note that $\Omega_{B/A} = k[x]/(nx^{n-1})$. If the characteristic of k does not divide n, then f is étale at $(x - a)$ for $a \neq 0$. In any case f is not étale at (x).

Example 2.7 (Artin–Scheier covering)**.** Let k be a field of positive characteristic p. Then

$$f : \operatorname{Spec} A[z]/(z^p - z - g(y)) \to \operatorname{Spec} k[y]$$

is finite étale, where $g(y) \in k[y]$. This implies that $\pi_1^{\text{ét}}(\mathbb{A}^1_k) \neq 0$.

Remark 2.8. Some literature defines unramified morphisms to be locally of finite type and $\Omega_{X/Y} = 0$, which is slightly weaker than our definition. There is no difference when X, Y are noetherian. The standard technique here is passing to limit. Assume that $f : X = \operatorname{Spec} B \to Y = \operatorname{Spec} A$ is of finite presentation. Write $B = A[x_1, \ldots, x_n]/I$ with $I = (g_1, \ldots, g_m)$. We can write $A = \operatorname{colim}_\lambda A_\lambda$ where $A_\lambda \subset A$ are finitely generated $\mathbb{Z}$-algebras, which are noetherian. Take λ large enought such that $g_1, \ldots, g_m \in A_\lambda[x_1, \ldots, x_n]$. Let $B_\lambda = A_\lambda[x_1, \ldots, x_m]/(g_1, \ldots, g_m)$. Then $B_\lambda \otimes_{A_\lambda} A \simeq B$ by construction. In particular, $f : X \to Y$ can be written as a base-change of $X_\lambda = \operatorname{Spec} B_\lambda \to Y_\lambda = \operatorname{Spec} A_\lambda$ which is of finite type over $\mathbb{Z}$. One can show that f is unramified (or, flat, étale) if and only if for λ large enough, f_λ is unramified (or, flat, étale). Therefore often properties of unramified/étale morphisms are reduced to the noetherian case.

Lemma 2.9. Étale morphisms are open maps.

Proof. See the above remark, and use the fact that flat of finite type morphisms between noetherian schemes are open. □