Concentration Inequalities

Tianjiao Nie

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1 Basic Inequalities

Lemma 1.1 (Markov)**.** Let $X \geq 0$ be a random variable with $\mathbb{E}[X] = \mu$. Then for $t > 0$,

$$\Pr[X \geq t\mu] \leq 1/t.$$

Proof. This follows from $\mathbb{E}[X] \geq \Pr[X \geq t] \cdot t + \Pr[X < t] \cdot 0 = t \Pr[X \geq t]$. □

Lemma 1.2 (Chebyshev)**.** Let X be a random variable with $\mathbb{E}[X] = \mu$ and $\mathrm{Var}[X] = \sigma^2$. Then for $t > 0$,

$$\Pr\big[|X - \mu| \geq t\sigma\big] \leq 1/t^2.$$

Proof. Note that $\Pr\big[|X - \mu| \geq t\sigma\big] = \Pr\big[|X - \mu|^2 \geq t^2\sigma^2\big]$ and then we apply Markov's inequality. □

2 The Chernoff Bound

3 The Berry–Esseen Theorem

Lemma 3.1 (Berry–Esseen)**.** Let $X_1, \ldots, X_m$ be i.i.d. random variables such that $\mathbb{E}[X_i] = 0$ and $\sum_{i=1}^m \mathrm{Var}[X_i] = 1$. Let $S = X_1 + \cdots + X_m$. We have $\mathbb{E}[S] = 0$ and $\mathrm{Var}[S] = 1$. Then for every $u \in \mathbb{R}$,

$$\big|\Pr[S \leq u] - \Pr_{Z \sim \mathcal{N}(0,1)}[Z \leq u]\big| \leq C\beta$$

where $C > 0$ is a universal constant and $\beta = \sum_{i=1}^m \mathbb{E}\big[|X_i|^3\big]$ is the third momentum.

Example 3.2. Consider $2m$ fair coins. Then

$$\Pr[\#H = \#T] = \frac{1}{2^{2m}}\binom{2m}{m} = \frac{(2m)!}{m!m!2^{2m}} \sim \frac{\sqrt{2\pi \cdot 2m}(2m/e)^{2m}}{\big[\sqrt{2\pi m}(m/e)^m\big]^2 2^{2m}} = \frac{1}{\sqrt{\pi m}}$$

where we used Stirling's approximation to estimate factorials.